

# ENTROPY ALGEBRAS AND BIRKHOFF FACTORIZATION

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**ABSTRACT.** We develop notions of Rota–Baxter structures and associated Birkhoff factorizations, in the context of min-plus semirings and their thermodynamic deformations, including deformations arising from quantum information measures such as the von Neumann entropy. We consider examples related to Manin’s renormalization and computation program, to Markov random fields and to counting functions and zeta functions of algebraic varieties.

## 1. INTRODUCTION

This paper is motivated by two different sources: Manin’s “renormalization and computation” program, [15], [16], [17], and the theory of “thermodynamic semirings” developed in [6], [19]. Manin proposed the use of an algebraic framework modeled on the Connes–Kreimer theory of renormalization [7] to achieve a renormalization of infinities that arise in computation (halting problem). In this formalism the Hopf algebra of Feynman graphs is replaced by a Hopf algebra of flow charts computing recursive functions. He suggested that natural characters of this Hopf algebra, of relevance to the computational setting, such as memory size or computing time, would be naturally taking values in a min-plus or max-plus (tropical) algebra instead of taking values in a commutative Rota–Baxter algebra, as in the case of renormalization in quantum field theory. Thus, in [15] he asked for an extension of the algebraic renormalization method based on Rota–Baxter algebras ([10], [11]) to tropical semirings. On the other hand, min-plus semirings admit deformations based on thermodynamic information measures, such as the Shannon entropy and generalizations, [19]. These are closely related to Maslov dequantization, [24].

In this paper we develop a unified approach to Rota–Baxter structures and Birkhoff factorizations in min-plus semirings and their thermodynamic deformations. In Section §2 we recall the basic definitions and properties of thermodynamic semirings, and we describe generalizations defined as deformations of the (tropical) trace using the von Neumann entropy and other entropy measures in quantum information. In §3 we introduce Rota–Baxter structures on min-plus semirings and we obtain a Birkhoff factorization of min-plus characters for Rota–Baxter structures of weight  $+1$  (unlike the original renormalization case that uses weight  $-1$  Rota–Baxter operators). In §4, we introduce Rota–Baxter structures on thermodynamic semirings and we relate them to Rota–Baxter structures on ordinary commutative rings. We construct Birkhoff factorizations in thermodynamic semirings with Rota–Baxter operators of weight  $+1$ . In §5 we extend the thermodynamic Rota–Baxter structures to the case of the von Neumann entropy and the trace deformation. In §6 we consider some explicit examples of Rota–Baxter operators of weight  $+1$  on commutative rings and on thermodynamic semirings, and we show that they determine Rota–Baxter structures of the same weight on Witt rings. We discuss some applications to zeta functions of algebraic varieties, seen as elements of Witt rings, as in [23]. In §7 we consider Rota–Baxter operators of weight  $-1$  on min-plus semirings, and we show that, under an additional superadditivity condition, one can still obtain Birkhoff factorizations. We also consider a variant of the construction, where the Birkhoff factorization is obtained from a pair of Rota–Baxter operators of weight  $-1$ , generalizing the pair  $T, id - T$  of the classical renormalization case. In §8 we consider three explicit examples of min-plus characters, motivated, respectively, by Manin’s renormalization and computation proposal [15], and the complexity theory of recursive functions [5]; by the theory of Markov random fields and Gibbs

states on graphs, [22]; and by the question of polynomial countability for the graph hypersurfaces of quantum field theory, [18].

## 2. THERMODYNAMIC SEMIRINGS AND OTHER THERMODYNAMIC DEFORMATIONS

After recalling the notion of thermodynamic semirings from [19], we introduce thermodynamic deformations of the trace, which extend the deformed addition of thermodynamic semirings from classical to quantum information. In particular, we interpret the case based on the von Neumann entropy as a Helmholtz free energy. We also discuss briefly functionals obtained as thermodynamic deformations of the integral, defined through the data of a dynamical system and its metric and topological entropies.

**2.1. Thermodynamic semirings.** The min-plus (or tropical) semiring  $\mathbb{T}$  is  $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ , with the operations  $\oplus$  and  $\odot$  given by

$$x \oplus y = \min\{x, y\},$$

with  $\infty$  the identity element for  $\oplus$  and with

$$x \odot y = x + y,$$

with 0 the identity element for  $\odot$ . The operations  $\oplus$  and  $\odot$  satisfy associativity and commutativity and distributivity of the product  $\odot$  over the sum  $\oplus$ .

We will occasionally consider also the analogous max-plus version  $\mathbb{T}_{\max} = \mathbb{R} \cup \{-\infty\}$ , with  $\oplus = \max$  and  $\odot = +$ . We will write  $\mathbb{T}_{\max}$ , when needed, to distinguish it from  $\mathbb{T} = \mathbb{T}_{\min}$ .

A notion of *thermodynamic semiring* was developed in [19], generalizing a construction of [6], as a deformation of the min-plus algebra, where the product  $\odot$  is unchanged, but the sum  $\oplus$  is deformed to a new operation  $\oplus_{\beta, S}$ , according to a binary entropy functional  $S$  and a deformation parameter  $\beta \geq 0$ , which we interpret thermodynamically as an inverse temperature (up to the Boltzmann constant which we set equal to 1). At zero temperature (that is,  $\beta \rightarrow \infty$ ) one recovers the unperturbed idempotent addition. The case where the entropy functional  $S$  is the Shannon entropy was considered in [6], in relation to geometry over the field with one element, while other entropy functionals, such as Rényi entropy or Tsallis entropy or Kullback–Leibler divergence are considered in [19], along with a general operadic formulation.

More precisely, for a fixed  $\beta \geq 0$  and a given entropy functional  $S$ , one defines on  $\mathbb{R} \cup \{\infty\}$  the operation

$$(2.1) \quad x \oplus_{\beta, S} y = \min_p \{px + (1-p)y - \frac{1}{\beta} S(p)\}.$$

The algebraic properties (commutativity, left and right identity, associativity) of this operation correspond to properties of the entropy functional (symmetry  $S(p) = S(1-p)$ , minima  $S(0) = S(1) = 0$ , and extensivity  $S(pq) + (1-pq)S(p/(1-pq)) = S(p) + pS(q)$ ). Thus, by the Khinchin axioms, imposing that all the algebraic properties of  $\mathbb{T}$  are preserved in the deformation singles out the Shannon entropy among the possible functionals  $S$ , while non-extensive entropy (see [12]) can be modeled by non-associative thermodynamic semirings. We refer the reader to [19] for more details.

When  $S$  is the Shannon entropy, the idempotent property  $x \oplus x = \min\{x, x\} = x$  of the tropical addition becomes in the deformed case  $x \oplus_{\beta, S} x = x - \beta^{-1} \log 2$ . This is immediately evident from  $x \oplus_{\beta, S} y = \min_p \{px + (1-p)y - \beta^{-1} S(p)\}$ , which for  $y = x$  gives  $x \oplus_{\beta, S} x = x - \beta^{-1} \max_p S(p) = x - \beta^{-1} \log 2$ . Moreover, in the case of the Shannon entropy, the deformed addition can be written equivalently as

$$(2.2) \quad x \oplus_{\beta, S} y = -\beta^{-1} \log \left( e^{-\beta x} + e^{-\beta y} \right).$$

The theory of thermodynamic semirings developed in [19] leads to a more general operadic and categorical formulation of entropy functionals (see §10 of [19]), which is similar in spirit to the approach of [3].

As in §10 of [19], consider a collection  $\mathcal{S} = \{S_n\}_{n \in \mathbb{N}}$  of  $n$ -ary entropy functionals  $S_n$ , satisfying the coherence condition

$$S_n(p_1, \dots, p_n) = S_m(p_{i_1}, \dots, p_{i_m}),$$

whenever, for some  $m < n$ , we have  $p_j = 0$  for all  $j \notin \{i_1, \dots, i_m\}$ .

Shannon, Rényi, Tsallis entropies satisfy the coherence condition, and so do, more generally, entropy functionals depending on functions  $f$  and  $g$  of the form

$$S_n(p_1, \dots, p_n) = f\left(\sum_{i=1}^n g(p_i)\right).$$

A collection  $\mathcal{S} = \{S_n\}_{n \in \mathbb{N}}$  as above determines a family of  $n$ -ary operations  $C_{n,\beta,\mathcal{S}}$  on  $\mathbb{R} \cup \{\infty\}$ ,

$$(2.3) \quad C_{n,\beta,\mathcal{S}}(x_1, \dots, x_n) = \min_p \left\{ \sum_{i=1}^n p_i x_i - \frac{1}{\beta} S_n(p_1, \dots, p_n) \right\},$$

where the minimum is taken over  $p = (p_i)$ , with  $\sum_i p_i = 1$ . More generally, given  $\mathcal{S}$  as above and the collection of all rooted tree  $\mathcal{T}$  with  $n$  leaves, and with fixed planar embeddings, we obtain  $n$ -ary operations  $C_{n,\beta,\mathcal{S},\mathcal{T}}(x_1, \dots, x_n)$  on  $\mathbb{R} \cup \{\infty\}$ , determined by the tree  $\mathcal{T}$  and the collection of entropy functionals  $S_j$  for  $j = 2, \dots, n+1$ . Namely, one defines  $C_{n,\beta,\mathcal{S},\mathcal{T}}(x_1, \dots, x_n)$  as the output of the tree  $\mathcal{T}$  with inputs  $x_1, \dots, x_n$  at the leaves and with an operation  $C_{m,\beta,\mathcal{S}}$  at each vertex of valence  $m+1$ . As shown in Theorem 10.9 of [19], these operations can be written equivalently as

$$(2.4) \quad C_{n,\beta,\mathcal{S},\mathcal{T}}(x_1, \dots, x_n) = \min_p \left\{ \sum_{i=1}^n p_i x_i - \frac{1}{\beta} S_{\mathcal{T}}(p_1, \dots, p_n) \right\},$$

with the  $S_{\mathcal{T}}(p_1, \dots, p_n)$  obtained from the  $S_j$ , for  $j = 2, \dots, n$ .

The data  $(\mathbb{T}, \mathcal{S})$  with  $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$  and with  $\mathcal{S} = \{S_n\}_{n \in \mathbb{N}}$  a coherent family of entropy functionals define an *information algebra*, which is an algebra over the  $A_\infty$ -operad of rooted trees, see §10 of [19].

**2.2. Von Neumann entropy and deformed traces.** When passing from classical to quantum information, probabilities  $P = (p_i)_{i=1}^n$  with  $p_i \geq 0$  and  $\sum_i p_i = 1$  are replaced by density matrices  $\rho$  with  $\rho^* = \rho$ ,  $\rho \geq 0$ , and  $\text{Tr}(\rho) = 1$ . The classical case is recovered as the case of diagonal matrices. Correspondingly, the entropy functionals, such as Shannon entropy, Rényi and Tsallis entropies, Kullback–Leibler relative entropy, have quantum information analogs, given by the von Neumann entropy and its generalizations. The algebraic structure of thermodynamic semirings, which encodes the axiomatic properties of classical entropy functionals, also generalizes to quantum information, no longer in the form of a deformed addition on a semiring, but as a deformed trace, as we discuss below.

For  $N \geq 1$ , let

$$\mathcal{M}^{(N)} = \{\rho \in M_{N \times N}(\mathbb{C}) \mid \rho^* = \rho, \rho \geq 0, \text{Tr}(\rho) = 1\}$$

be the convex set of density matrices. The von Neumann entropy

$$(2.5) \quad \mathcal{N}(\rho) = -\text{Tr}(\rho \log \rho), \quad \text{for } \rho \in \mathcal{M}^{(N)},$$

is the natural generalization of the Shannon entropy to the quantum information setting. It reduces to the Shannon entropy in the diagonal case.

As above, let  $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$  be the tropical min-plus semiring. Let  $M_{N \times N}(\mathbb{T})$  denote  $N \times N$ -matrices with entries in  $\mathbb{R} \cup \{\infty\}$ , with the operations of idempotent matrix addition and multiplication

$$(A \oplus B)_{ij} = \min\{A_{ij}, B_{ij}\}, \quad (A \odot B)_{ij} = \oplus_k A_{ik} \odot B_{kj} = \min_k \{A_{ik} + B_{kj}\}.$$

The trace is defined as:

$$(2.6) \quad \text{Tr}^\oplus(A) = \min_i \{A_{ii}\}.$$

We also denote by

$$(2.7) \quad \widetilde{\text{Tr}}^\oplus(A) := \min_{U \in U(N)} \min_i \{(UAU^*)_{ii}\} \leq \text{Tr}^\oplus(A).$$

We introduce thermodynamic deformations of the trace, by setting

$$(2.8) \quad \text{Tr}_{\beta, S}^\oplus(A) := \min_{\rho \in \mathcal{M}^{(N)}} \{\text{Tr}(\rho A) - \beta^{-1} S(\rho)\},$$

where  $\text{Tr}$  in the right-hand-side is the ordinary trace, with  $\text{Tr}(\rho A) = \langle A \rangle$  the expectation value of the observable  $A$  with respect to the state  $\varphi(\cdot) = \text{Tr}(\rho \cdot)$ .

**Lemma 2.1.** *The zero temperature ( $\beta \rightarrow \infty$ ) limit of (2.8) gives*

$$(2.9) \quad \lim_{\beta \rightarrow \infty} \text{Tr}_{\beta, S}^\oplus(A) = \widetilde{\text{Tr}}^\oplus(A).$$

*Proof.* We can identify  $\mathcal{M}^{(N)} = \cup_{U \in U(N)} U \cdot \Delta_{N-1}$ , with the simplex  $\Delta_{N-1} = \{P = (p_i)_{i=1}^N \mid p_i \geq 0, \sum_i p_i = 1\}$ , where the action of  $U \in U(N)$  is by  $P \mapsto U \cdot P := U^* P U$ , where  $P$  is identified with the diagonal density matrix with diagonal entries  $p_i$ . We then have

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \text{Tr}_{\beta, S}^\oplus(A) &= \min_{\rho \in \mathcal{M}^{(N)}} \{\text{Tr}(\rho A)\} = \min_{P=(p_i) \in \Delta_{N-1}} \{\text{Tr}(P U A U^*)\} \\ &= \min_{P=(p_i) \in \Delta_{N-1}} \left\{ \sum_i p_i (U A U^*)_{ii} \right\} = \min_i \{(U A U^*)_{ii}\}. \end{aligned}$$

□

Recall that, for  $\rho, \sigma \in \mathcal{M}^{(N)}$ , the quantum relative entropy is defined as

$$(2.10) \quad S(\rho \parallel \sigma) = \text{Tr}(\rho(\log \rho - \log \sigma)).$$

**Lemma 2.2.** *When  $A = A^*$  with  $A \geq 0$ , the expression  $\text{Tr}(\rho A) - \beta^{-1} \mathcal{N}(\rho)$  can be identified with a relative entropy*

$$(2.11) \quad \text{Tr}(\rho A) - \beta^{-1} \mathcal{N}(\rho) = \frac{1}{\beta} S(\rho \parallel \sigma_{\beta, A}) - \frac{1}{\beta} \log Z_A(\beta),$$

where

$$\sigma_{\beta, A} = \frac{e^{-\beta A}}{Z_A(\beta)}, \quad \text{with} \quad Z_A(\beta) = \text{Tr}(e^{-\beta A}).$$

*Proof.* This follows by simply writing

$$\text{Tr}(\rho A) + \beta^{-1} \text{Tr}(\rho \log \rho) = \beta^{-1} \text{Tr}(\rho(\log \rho - \log e^{-\beta A})).$$

□

**Proposition 2.3.** *When  $A = A^*$  with  $A \geq 0$ , the deformed trace (2.8) with  $S = \mathcal{N}$  the von Neumann entropy is given by*

$$(2.12) \quad \text{Tr}_{\beta, \mathcal{N}}^\oplus(A) = -\frac{\log Z_A(\beta)}{\beta},$$

with  $Z_A(\beta) = \text{Tr}(e^{-\beta A})$ .

*Proof.* By the previous lemma, we have

$$\mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A) = \min_{\rho \in \mathcal{M}^{(N)}} \frac{1}{\beta} S(\rho || \sigma_{\beta, A}) - \frac{1}{\beta} \log Z_A(\beta)$$

The relative entropy have the property that  $S(\rho || \sigma) \geq 0$  with minimum at  $\rho = \sigma$  where  $S(\rho || \rho) = 0$ . Thus, the minimum of the above expression is  $\beta^{-1} \log Z_A(\beta)$ .  $\square$

**Remark 2.4.** The expression  $\beta^{-1} \log Z_A(\beta)$  can be interpreted as the Helmholtz free energy in quantum statistical mechanics.

**Remark 2.5.** In the case where  $A$  is the  $2 \times 2$  diagonal matrix with diagonal entries  $(x, y)$ , the expression  $\mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A) = \beta^{-1} \log Z_A(\beta)$  recovers the usual deformed addition  $x \oplus_{\beta, S} y$  in the thermodynamic semiring with  $S$  the Shannon entropy, in the form (2.2),

$$x \oplus_{\beta, S} y = \min_p \{px + (1-p)y - \beta^{-1} S(p)\} = -\beta^{-1} \log(e^{-\beta x} + e^{-\beta y}).$$

**Corollary 2.6.** When  $A = A^*$  with  $A \geq 0$ , the zero temperature limit is

$$\widetilde{\mathrm{Tr}}^{\oplus}(A) = \min\{\lambda \in \mathrm{Spec}(A)\}.$$

*Proof.* We take the limit as  $\beta \rightarrow \infty$  of  $\mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A) = \beta^{-1} \log Z_A(\beta)$ . The leading term is given by  $\beta^{-1} \log e^{-\beta \lambda_{\min}}$ , where  $\lambda_{\min} = \min\{\lambda \in \mathrm{Spec}(A)\}$ , hence comparing with Lemma 2.1, we get

$$\min_{U \in U(N)} \min_i \{(UAU^*)_{ii}\} = \lambda_{\min}.$$

$\square$

In the following, we will use the unconventional symbol  $\boxplus$  for the direct sum of matrices, to distinguish it from the symbol  $\oplus$  that we have adopted for the addition operation in min-plus semirings. The deformed trace has following behavior.

**Proposition 2.7.** For a matrix  $A^* = A$ ,  $A \geq 0$ , that is a direct sum of two matrices  $A = A_1 \boxplus A_2$  with  $A_i = A_i^*$  and  $A_i \geq 0$ , the deformed traces satisfy

$$(2.13) \quad \mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A) = \mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A_1) \odot \mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A_2),$$

where  $\odot$  is the product in the tropical semiring  $\mathbb{T}$ .

*Proof.* By Proposition 2.3 we have  $\mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A) = -\beta^{-1} \log \mathrm{Tr}(e^{-\beta A})$ . For  $A$  a direct sum of  $A_1$  and  $A_2$  we have  $e^{-\beta A} = e^{-\beta A_1} \otimes e^{-\beta A_2}$  and  $\mathrm{Tr}(e^{-\beta A_1} \otimes e^{-\beta A_2}) = \mathrm{Tr}(e^{-\beta A_1}) \mathrm{Tr}(e^{-\beta A_2})$ , hence we get

$$\mathrm{Tr}_{\beta, \mathcal{N}}^{\oplus}(A) = -\beta^{-1} \left( \log \mathrm{Tr}(e^{-\beta A_1}) + \log \mathrm{Tr}(e^{-\beta A_2}) \right).$$

$\square$

**2.3. Generalizations of von Neumann entropy and relative entropy.** In addition to the von Neumann entropy, there are several other natural entropy functionals in quantum information. Some of the main examples (see [4], [25]) are

- The quantum relative entropy: for  $\rho, \sigma \in \mathcal{M}^{(N)}$

$$S(\rho || \sigma) = \mathrm{Tr}(\rho(\log \rho - \log \sigma)).$$

- The quantum Rényi entropy: for  $\rho \in \mathcal{M}^{(N)}$

$$S_q(\rho) = \frac{1}{1-q} \log \mathrm{Tr}(\rho^q).$$

- The Belavkin–Staszewski relative entropy: for  $\rho, \sigma \in \mathcal{M}^{(N)}$

$$S_{BS}(\rho || \sigma) = \mathrm{Tr}(\rho \log(\rho^{1/2} \sigma^{-1} \rho^{1/2})).$$

- The quantum Tsallis entropy: for  $\rho \in \mathcal{M}^{(N)}$

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \text{Tr}(\rho(\rho^{\alpha-1} - I)).$$

- The Umegaki deformed relative entropy: for  $\rho, \sigma \in \mathcal{M}^{(N)}$

$$S_\alpha(\rho||\sigma) = \frac{4}{1-\alpha^2} \text{Tr}((I - \sigma^{(\alpha+1)/2} \rho^{(\alpha-1)/2}) \rho).$$

The fact that there is a large supply of entropies and relative entropies in the quantum case depends on the fact that the expression  $\rho\sigma^{-1}$ , in going from classical to quantum, can be replaced by several different expressions, when  $\rho$  and  $\sigma$  do not commute. All of these entropy functionals give rise to corresponding thermodynamic deformations  $\text{Tr}_{\beta,S}^\oplus(A)$  of the tropical trace  $\text{Tr}^\oplus(A)$ , defined as in (2.8). In the case of relative entropies, we assume given a fixed density matrix  $\sigma$  and we set  $S_\sigma(\rho) = S(\rho||\sigma)$ , so that

$$\text{Tr}_{\beta,S_\sigma}^\oplus(A) = \min_{\rho \in \mathcal{M}^{(N)}} \{ \text{Tr}(\rho A) - \beta^{-1} S(\rho||\sigma) \}.$$

This is the natural generalization of the case of thermodynamic semirings with  $S$  the Kullback–Leibler relative entropy, discussed in [19].

**2.4. Thermodynamically deformed states on  $C^*$ -algebras.** The construction discussed above using entropy functionals on matrix algebras can be extended to a more general setting of  $C^*$ -algebras. Let  $\mathcal{A}$  be a unital separable  $C^*$ -algebra (noncommutative in general) and let  $\mathcal{M}$  be the convex set of states on  $\mathcal{A}$ , namely continuous linear functionals  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$  that are normalized by  $\varphi(1) = 1$  and satisfy the positivity condition  $\varphi(a^*a) \geq 0$ , for all  $a \in \mathcal{A}$ .

There is a general notion of relative entropy  $S(\varphi||\psi)$  of states  $\varphi, \psi \in \mathcal{M}$  on a  $C^*$ -algebra  $\mathcal{A}$ , see §2.3 of [20], with the same bi-convexity property of the usual relative entropy, and with  $S(\varphi||\psi) \geq 0$  for all states  $\varphi, \psi$ , with equality attained only when  $\varphi = \psi$ . In the case where there is a trace  $\tau : \mathcal{A} \rightarrow \mathbb{C}$  on the  $C^*$ -algebra, consider states of the form  $\varphi(a) = \tau(a\xi)$ ,  $\psi(a) = \tau(a\eta)$ , where  $\xi, \eta$  are positive elements in the algebra with  $\tau(\xi) = \tau(\eta) = 1$ . Then the relative entropy reduces to

$$(2.14) \quad S(\varphi||\psi) = \tau(\xi(\log \xi - \log \eta)).$$

Given a state  $\psi \in \mathcal{M}$ , we define its thermodynamical deformation  $\psi_{\beta,S}$  as

$$(2.15) \quad \psi_{\beta,S}(a) = \min_{\varphi \in \mathcal{M}} \{ \varphi(a) + \beta^{-1} S(\varphi||\psi) \}.$$

Notice that, in the finite dimensional case of a matrix algebra, this agrees with our previous definition of the deformation of the trace, since states are of the form  $\varphi(a) = \text{Tr}(a\rho)$  for some density matrix  $\rho$  and the von Neumann entropy can be seen as  $\mathcal{N}(\rho) = -S(\varphi||\psi)$ , for  $\varphi(a) = \text{Tr}(a\rho)$  and  $\psi(a) = \text{Tr}(a)$ .

While this more general setting is not the main focus of the present paper, we illustrate the construction in one significant example. Let  $\mathcal{A}_\theta$  be the irrational rotation algebra (noncommutative torus) with unitary generators  $U, V$  satisfying  $VU = e^{2\pi i\theta}UV$ . Let  $\tau$  be the canonical trace,  $\tau(U^n V^m) = 0$  for  $(n, m) \neq (0, 0)$  and  $\tau(1) = 1$ . We consider only states of the form  $\varphi(a) = \tau(a\xi)$  for some positive element  $\xi \in \mathcal{A}_\theta$ . Let  $\mathcal{M}_\tau$  be the set of such states. We then consider the thermodynamic deformation of the canonical trace given by

$$(2.16) \quad \tau_{\beta,S}(a) = \min_{\varphi \in \mathcal{M}_\tau} \{ \varphi(a) + \beta^{-1} S(\varphi||\tau) \}.$$

We then obtain the following result, whose proof is completely analogous to Proposition 2.3 above.

**Proposition 2.8.** *For  $a = h^*h \geq 0$  in  $\mathcal{A}_\theta$ , the deformed trace  $\tau_{\beta,S}(a)$  is given by*

$$\tau_{\beta,S}(a) = \min_{\varphi \in \mathcal{M}_\tau} \{ \beta^{-1} S(\varphi || \varphi_{\beta,a}) - \beta^{-1} \log \tau(e^{-\beta a}) \} = -\beta^{-1} \log \tau(e^{-\beta a})$$

for the KMS $_\beta$  state

$$\varphi_{\beta,a}(b) = \frac{\tau(be^{-\beta a})}{\tau(e^{\beta a})}$$

of the time evolution  $\sigma_t(b) = e^{ita} b e^{-ita}$  on  $\mathcal{A}_\theta$ , with  $-\beta^{-1} \log \tau(e^{-\beta a})$  the associated the Helmholtz free energy.

The limit  $\lim_{\beta \rightarrow \infty} \tau_{\beta,S}(a)$  should then be regarded as a notion of “tropicalization” of the von Neumann trace  $\tau$  of the noncommutative torus.

**2.5. Thermodynamic deformations and entropy of dynamical systems.** We consider a locally compact Hausdorff space  $X$ , with a dynamical system  $\sigma : X \rightarrow X$ . We focus in particular on the case where  $X$  is a Cantor set, identified with the set of infinite words  $w = w_0 w_1 \dots w_i w_{i+1} \dots$  in a finite alphabet  $w_i \in \mathfrak{A}$ , with  $\#\mathfrak{A} = n$ , with the topology generated by cylinder sets  $\mathcal{C}(a_0, \dots, a_N) = \{w \in X \mid w_i = a_i, 0 \leq i \leq N\}$ . Let  $d(x, y)$  be a compatible metric. As dynamical system, we consider in particular the case of the one-sided shift  $\sigma : X \rightarrow X$ , defined by  $\sigma(w)_i = w_{i+1}$ .

A Bernoulli measure  $\mu_P$  on  $X$  is a shift-invariant measure defined by a probability  $P = (p_1, \dots, p_n)$ , with  $p_i \geq 0$  and  $\sum_{i=1}^n p_i = 1$ , on the alphabet  $\mathfrak{A}$ . It assigns measure  $\mu_P(\mathcal{C}(a_0, \dots, a_N)) = p_{a_0} \cdots p_{a_N}$  to the cylinder sets.

A Markov measure  $\mu_{P,\rho}$  on  $X$  is a shift-invariant measure defined by a pair  $(P, \rho)$  of a probability  $P = (p_1, \dots, p_n)$  on  $\mathfrak{A}$  and a stochastic matrix  $\rho$  satisfying  $P\rho = P$ . It assigns measure  $\mu_{P,\rho}(\mathcal{C}(a_0, \dots, a_N)) = p_{a_0} \rho_{a_0 a_1} \cdots \rho_{a_{N-1} a_N}$ . A Markov measure  $\mu_{P,\rho}$  is supported on a subshift of finite type  $X_A \subset X$ , given by  $X_A = \{w \in X \mid A_{w_i w_{i+1}} = 1, \forall i \geq 0\}$ , where the matrix  $A_{ij}$  has entries 0 or 1, according to whether the corresponding entry  $\rho_{ij}$  of the stochastic matrix  $\rho$  is  $\rho_{ij} = 0$  or  $\rho_{ij} \neq 0$ . The subspace  $X_A$  is shift-invariant.

Recall that, for  $\mu$  a  $\sigma$ -invariant probability measure on  $X$ , one defines the entropy  $S(\mu, \sigma)$  as the  $\mu$ -almost everywhere value of the local entropy

$$h_{\mu,\sigma}(x) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_\sigma(x, n, \delta)),$$

where  $B_\sigma(x, n, \delta) = \{y \in X \mid d(\sigma^j(x), \sigma^j(y)) < \delta, \forall 0 \leq j \leq n\}$  are the Bowen balls, see [21]. In the case of a Bernoulli measure  $\mu = \mu_P$ , the dynamical entropy agrees with the Shannon entropy of  $P$ ,

$$S(\mu_P, \sigma) = -\sum_{i=1}^N p_i \log p_i,$$

while for a Markov measure, the dynamical entropy is

$$S(\mu_{P,\rho}, \sigma) = -\sum_{i=1}^N p_i \sum_{j=1}^N \rho_{ij} \log \rho_{ij}.$$

In the same spirit as the thermodynamic deformations of the trace discussed previously in this section, we can introduce thermodynamic deformations of the integral of functions  $f \in C(X, \mathbb{R})$  by setting

$$(2.17) \quad \int_X^{(\beta,S)} f(x) dx := \inf_{\mu} \left\{ \int_X f(x) d\mu(x) - \beta^{-1} S(\mu, \sigma) \right\},$$



where the infimum is taken over a specific class of  $\sigma$ -invariant measures, for example over all Bernoulli measures, or over all Markov measures, or more generally over all  $\sigma$ -invariant ergodic measures. In the latter case, recall that the topological entropy of the shift  $\sigma$  is

$$h(X, \sigma) = \sup_{\mu} \{S(\mu, \sigma)\},$$

with the supremum taken over all  $\sigma$ -invariant ergodic measures.

We will not discuss further the properties of the functionals (2.17), as that would lead us outside the main scope of the present paper.

### 3. ROTA–BAXTER STRUCTURES AND BIRKHOFF FACTORIZATION IN MIN-PLUS SEMIRINGS

A mathematical model of renormalization for perturbative quantum field theories, based on a commutative Hopf algebra  $\mathcal{H}$ , a Rota–Baxter algebra  $\mathcal{R}$  and the Birkhoff factorization of morphisms of commutative algebras  $\phi : \mathcal{H} \rightarrow \mathcal{R}$ , was developed in [7], [10], [11]. More recently, an approach to the theory of computation and the halting problem modelled on quantum field theory and renormalization was developed in [15], [16], [17], and further investigated in [9]. In view of applications to the theory of computation, it was observed in §4.6 of [15] that it would be useful to replace characters given by commutative algebra homomorphisms  $\phi : \mathcal{H} \rightarrow \mathcal{R}$  from the Hopf algebra to a Rota–Baxter algebra, with characters  $\psi : \mathcal{H} \rightarrow \mathbb{S}$  with values in a min-plus semiring, satisfying  $\psi(xy) = \psi(x) + \psi(y) = \psi(x) \odot \psi(y)$ . With this motivation in mind, we develop here a setting for Rota–Baxter structures and Birkhoff factorization taking place in min-plus semirings and in their thermodynamic deformations.

**3.1. Rota–Baxter algebras and renormalization.** We refer the reader to [13] for a general introduction to the subject of Rota–Baxter algebras. For their use in renormalization of perturbative quantum field theories, we refer the reader to [8], [11], [18], for more details.

A Rota–Baxter algebra (ring) of weight  $\lambda$  is a unital commutative algebra (ring)  $\mathcal{R}$  endowed with a linear operator  $\mathcal{T} : \mathcal{R} \rightarrow \mathcal{R}$  which satisfies the  $\lambda$ -Rota–Baxter identity

$$(3.1) \quad \mathcal{T}(a)\mathcal{T}(b) = \mathcal{T}(a\mathcal{T}(b)) + \mathcal{T}(\mathcal{T}(a)b) + \lambda\mathcal{T}(ab).$$

We will be especially interested in two cases, namely  $\lambda = \pm 1$ , which correspond, respectively, to the identities

$$(3.2) \quad \mathcal{T}(a)\mathcal{T}(b) = \mathcal{T}(a\mathcal{T}(b)) + \mathcal{T}(\mathcal{T}(a)b) + \mathcal{T}(ab),$$

$$(3.3) \quad \mathcal{T}(a)\mathcal{T}(b) + \mathcal{T}(ab) = \mathcal{T}(a\mathcal{T}(b)) + \mathcal{T}(\mathcal{T}(a)b).$$

The latter case, with weight  $\lambda = -1$ , is the one used in renormalization in quantum field theory, while we will see that the case  $\lambda = +1$  is more natural to adapt to the setting of min-plus semirings.

Laurent polynomials  $\mathcal{R} = \mathbb{C}[t, t^{-1}]$  with the projection  $\mathcal{T}$  onto the polar part are the prototype example of a Rota–Baxter algebra of weight  $-1$ . Recall also that, if  $\mathcal{T}$  is a Rota–Baxter operator of weight  $\lambda \neq 0$ , then  $\lambda^{-1}\mathcal{T}$  is a Rota–Baxter operator of weight 1.

When  $\lambda = -1$  the Rota–Baxter operator  $\mathcal{T}$  determines a decomposition of  $\mathcal{R}$  into two commutative algebras (rings),  $\mathcal{R}_+ = (1 - \mathcal{T})\mathcal{R}$  and  $\mathcal{R}_-$  given by the unitization of  $\mathcal{T}\mathcal{R}$ .

Algebraic renormalization is a factorization procedure for Hopf algebra characters. More precisely, one considers over a field or ring  $k$  a graded connected commutative Hopf algebra  $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$  with  $\mathcal{H}_0 = k$  and the set of homomorphisms of commutative rings  $\text{Hom}(\mathcal{H}, \mathcal{R})$ , where the target  $\mathcal{R}$  is a Rota–Baxter ring of weight  $\lambda = -1$ .



The convolution product  $\star$  of morphisms  $\phi_1, \phi_2 \in \text{Hom}(\mathcal{H}, \mathcal{R})$  is dual to the coproduct in  $\mathcal{H}$ , that is,

$$(3.4) \quad \phi_1 \star \phi_2(x) = \langle \phi_1 \otimes \phi_2, \Delta(x) \rangle = \sum \phi_1(x^{(1)})\phi_2(x^{(2)}),$$

where

$$\Delta(x) = \sum x^{(1)} \otimes x^{(2)} = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''.$$

The Birkhoff factorization of a morphism  $\phi \in \text{Hom}(\mathcal{H}, \mathcal{R})$  is a multiplicative decomposition

$$(3.5) \quad \phi = (\phi_- \circ S) \star \phi_+,$$

where  $S$  is the antipode, defined inductively by

$$S(x) = -x - \sum S(x')x'',$$

where  $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ , with the  $x', x''$  of lower degrees.

The two parts  $\phi_{\pm}$  of the Birkhoff factorization are morphisms of commutative algebras (rings)  $\phi_{\pm} : \mathcal{H} \rightarrow \mathcal{R}_{\pm}$ . The decomposition is obtained inductively through the explicit formula

$$(3.6) \quad \phi_-(x) = -\mathcal{T}(\phi(x) + \sum \phi_-(x')\phi(x'')) \quad \text{and} \quad \phi_+(x) = (1 - \mathcal{T})(\phi(x) + \sum \phi_-(x')\phi(x'')).$$

We denote by  $\tilde{\phi}(X)$  the Bogolyubov-Parashchuk “preparation” of  $\phi(X)$

$$(3.7) \quad \tilde{\phi}(x) := \phi(x) + \sum \phi_-(x')\phi(x'').$$

The fact that  $\phi_-$ , constructed inductively as above, is still a homomorphism of commutative rings  $\phi_- : \mathcal{H} \rightarrow \mathcal{R}_-$  is obtained by comparing

$$(3.8) \quad \phi_-(xy) = -\mathcal{T}(\tilde{\phi}(x)\tilde{\phi}(y)) + \mathcal{T}(\mathcal{T}(\tilde{\phi}(x))\tilde{\phi}(y) + \tilde{\phi}(x)\mathcal{T}(\tilde{\phi}(y)))$$

and

$$(3.9) \quad \phi_-(x)\phi_-(y) = \mathcal{T}(\tilde{\phi}(x))\mathcal{T}(\tilde{\phi}(y))$$

using the Rota–Baxter identity for  $\mathcal{T}$ . It then follows that  $\phi_+$  is also a ring homomorphism. The expression for  $\phi_-(xy)$  above is easily obtained by decomposing the terms  $(xy)'$  and  $(xy)''$  in the non-primitive part of the coproduct  $\Delta(xy)$  in terms of  $x, y, x'$  and  $x'', y'$  and  $y''$ . We will return to this argument below.

**3.2. Rings and semirings.** The usual setting of Rota–Baxter algebras recalled above is based on commutative rings  $\mathcal{R}$  with a linear operator  $\mathcal{T}$  satisfying the identity (3.1). Our purpose in this section is to extend this notion to min-plus semirings and their thermodynamic deformations and relate the Rota–Baxter structures and Birkhoff factorization on semirings to the ordinary ones on rings.

To this purpose, we will consider min-plus semirings, and thermodynamic deformations that are related to commutative ring via a “logarithm” map.

The kind of semirings we consider are semirings  $\mathbb{S}$  with min-plus operations  $\oplus, \otimes$ , for which thermodynamic deformations  $\mathbb{S}_{\beta, S}$  are defined, with  $S$  the Shannon entropy.

The Gelfand correspondence between compact Hausdorff spaces  $X$  and commutative unital  $C^*$ -algebras  $C(X)$  admits a generalizations for semirings of continuous functions  $C(X, \mathbb{T})$  with values in the tropical semiring  $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ , with the pointwise operations, see [14].

Thus, a large class of examples of semirings  $\mathbb{S}$  of the type described above is given by  $\mathbb{S} = C(X, \mathbb{T})$  with the pointwise  $\oplus = \min$  and  $\odot = +$  operations, and their thermodynamic deformations  $\mathbb{S}_{\beta, S} = C(X, \mathbb{T}_{\beta, S})$  with the pointwise deformed addition  $\oplus_{\beta, S}$ .

**Definition 3.1.** Let  $\mathcal{R}$  be a commutative ring (or algebra) and let  $\mathbb{S}$  be a semiring with min-plus operations  $\oplus, \otimes$ . The pair  $(\mathcal{R}, \mathbb{S})$  is a logarithmically related pair, if there is a bijective map  $\mathcal{L} : \text{Dom}(\mathcal{L}) \subset \mathcal{R} \rightarrow \mathbb{S}$  satisfying  $\mathcal{L}(ab) = \mathcal{L}(a) + \mathcal{L}(b) = \mathcal{L}(a) \odot \mathcal{L}(b)$ , for all  $a, b \in \text{Dom}(\mathcal{L})$ .

Let  $\mathbb{S}_{\beta, S}$  be a thermodynamic deformation of  $\mathbb{S}$ , for which we can write the deformed addition as

$$f_1 \oplus_{\beta, S} f_2 = -\beta^{-1} \log(E(-\beta f_1) + E(-\beta f_2)),$$

where  $E : \mathbb{S} \rightarrow \text{Dom}(\mathcal{L}) \subset \mathcal{R}$  denotes the inverse of  $\mathcal{L}$  and  $+$  is addition in the ring  $\mathcal{R}$ . The undeformed  $\odot$  operation in  $\mathbb{S}$  is related to the product in  $\mathcal{R}$  by  $f_1 + f_2 = \mathcal{L}(E(f_1)) + \mathcal{L}(E(f_2)) = \mathcal{L}(E(f_1)E(f_2))$ .

**Example 3.2.** The above applies to the case for all the semirings  $\mathbb{S}_{\beta, S} = C(X, \mathbb{T}_{\beta, S})$ , with the usual deformed addition, with  $S$  the Shannon entropy, given by

$$f_1 \oplus_{\beta, S} f_2 = -\beta^{-1} \log(e^{-\beta f_1} + e^{-\beta f_2}).$$

In this case  $\mathcal{R} = C(X, \mathbb{R})$  and the subset  $\text{Dom}(\mathcal{L}) \subset \mathcal{R}$  is given by functions  $a \in C(X, \mathbb{R}_+^*)$ , with  $a = e^{-\beta f}$ . In other words  $\mathcal{L}(a) = -\beta^{-1} \log(a)$ .

**Example 3.3.** Let  $\mathcal{R}$  be the ring of formal power series  $\mathbb{Q}[[t]]$  in one variable, with rational coefficients. Let  $\text{Dom}(\mathcal{L}) \subset \mathcal{R}$  be the subset of power series  $\alpha(t) = \sum_{k \geq 0} a_k t^k$  with  $a_0 = 1$ . Then  $\mathcal{L}$  is the formal logarithm  $\mathcal{L}(1 + \alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \alpha^k$ , mapping  $\mathcal{L} : \text{Dom}(\mathcal{L}) \rightarrow \mathbb{Q}[[t]]$ . It satisfies  $\mathcal{L}(\alpha\gamma) = \mathcal{L}(\alpha) + \mathcal{L}(\gamma)$ . The inverse of the formal logarithm  $\mathcal{L}$  is given by the formal exponential  $E(\gamma) = \sum_{k \geq 0} \gamma^k / k!$ . We can view  $\mathbb{Q}[[t]]$  as a thermodynamic semiring with deformed addition

$$\alpha_1 \oplus_{\beta, S} \alpha_2 = \beta^{-1} \mathcal{L}(E(-\beta \alpha_1) + E(-\beta \alpha_2)),$$

for  $\beta \in \mathbb{Q}$ , and undeformed multiplication  $\alpha_1 \odot \alpha_2 = \alpha_1 + \alpha_2$ .

For simplicity, in the following we will always write simply  $\log$  and  $\exp$  for the maps relating a ring  $\mathcal{R}$  and a semiring  $\mathbb{S}$ , as in definition 3.1.

### 3.3. Birkhoff factorization in min-plus semirings.

**Definition 3.4.** Let  $\mathbb{S}$  be a min-plus semiring as above. Let  $\mathcal{H}$  be a graded, connected, commutative Hopf algebra. A min-plus character (or  $\mathbb{S}$ -character) of the Hopf algebra is a map  $\psi : \mathcal{H} \rightarrow \mathbb{S}$  that satisfies the conditions  $\psi(1) = 0$  and

$$(3.10) \quad \psi(xy) = \psi(x) + \psi(y), \quad \forall x, y \in \mathcal{H}.$$

We also define a convolution product of min-plus characters. The intuition behind the definition comes from a standard heuristic reasoning, which regards the min-plus algebra as the “arithmetic of orders of magnitude”. Namely, when  $\epsilon \rightarrow 0$ , the leading term in  $\epsilon^\alpha + \epsilon^\beta$  is  $\epsilon^{\min\{\alpha, \beta\}}$ , while the leading term of  $\epsilon^\alpha \epsilon^\beta$  is  $\epsilon^{\alpha+\beta}$ . Thus, the notion of convolution product for min-plus characters should reflect the behavior of the leading order in the usual notion of convolution product of (commutative algebra valued) characters.

**Definition 3.5.** For  $\psi_1, \psi_2$  as above, the convolution product  $\psi_1 \star \psi_2$  is given by

$$(3.11) \quad (\psi_1 \star \psi_2)(x) = \min\{\psi_1(x^{(1)}) + \psi_2(x^{(2)})\} = \bigoplus (\psi_1(x^{(1)}) \odot \psi_2(x^{(2)})),$$

where the minimum is taken over all the pairs  $(x^{(1)}, x^{(2)})$  that appear in the coproduct  $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$  in the Hopf algebra  $\mathcal{H}$ , and  $\oplus = \min$  and  $\odot = +$  are the (pointwise) operations of the semiring  $\mathbb{S}$ .

Similarly, we reformulate the notion of Birkhoff factorization in the following way.

**Definition 3.6.** Let  $\psi$  be a min-plus character of the Hopf algebra  $\mathcal{H}$ . A Birkhoff factorization of  $\psi$  is a decomposition  $\psi_+ = \psi_- \star \psi$ , with  $\star$  the convolution product (3.11), where  $\psi_{\pm}$  satisfy (3.10).

Notice that, unlike the usual way of writing Birkhoff factorizations in the form (3.5), the formulation above as  $\psi_+ = \psi_- \star \psi$  does not require the use of the antipode of the Hopf algebra, hence it extends to the case where  $\mathcal{H}$  is a bialgebra. Since in our main applications  $\mathcal{H}$  will be a Hopf algebra, we maintain this assumption in the following.

**3.4. Rota–Baxter operators on min-plus semirings.** Let  $\mathbb{S}$  be a min-plus semiring, with (pointwise) operations  $\oplus$  and  $\odot$ . A map  $T : \mathbb{S} \rightarrow \mathbb{S}$  is  $\oplus$ -additive if it is monotone, namely  $T(a) \leq T(b)$  if  $a \leq b$ , for all  $a, b \in \mathbb{S}$ . For a semiring of the form  $\mathbb{S} = C(X, \mathbb{T})$  the condition is pointwise in  $t \in X$ .

We define Rota–Baxter structures with weight  $\lambda > 0$  as follows.

**Definition 3.7.** A min-plus semiring  $(\mathbb{S}, \oplus, \odot)$  is a Rota–Baxter semiring of weight  $\lambda > 0$  if there is a  $\oplus$ -additive map  $T : \mathbb{S} \rightarrow \mathbb{S}$ , which for all  $f_1, f_2 \in \mathbb{S}$  satisfies the identity

$$(3.12) \quad T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2)) \oplus T(f_1 \odot f_2) \odot \log \lambda.$$

Similarly, we can define Rota–Baxter structures of weight  $\lambda < 0$  in the following way.

**Definition 3.8.** A min-plus semiring  $(\mathbb{S}, \oplus, \odot)$  is a Rota–Baxter semiring of weight  $\lambda < 0$  if there is a  $\oplus$ -additive map  $T : \mathbb{S} \rightarrow \mathbb{S}$ , which for all  $f_1, f_2 \in \mathbb{S}$  satisfies the identity

$$(3.13) \quad T(f_1) \odot T(f_2) \oplus T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2)).$$

We have the following result on the existence of Birkhoff factorizations. As in the usual case, the proof is constructive, as it inductively defines the two parts of the factorization.

**Theorem 3.9.** Let  $\psi : \mathcal{H} \rightarrow \mathbb{S}$  be a min-plus character of a graded, connected, commutative Hopf algebra  $\mathcal{H}$ . Assume that the target semiring  $\mathbb{S}$  has a Rota–Baxter structure of weight  $+1$ , as in Definition 3.7. Then there is a Birkhoff factorization  $\psi_+ = \psi_- \star \psi$ , where  $\psi_-$  and  $\psi_+$  are also min-plus characters.

*Proof.* As in the usual Rota–Baxter algebra case, we construct the factors  $\psi_{\pm}$  inductively. We define the Bogolyubov–Parashchuk preparation of  $\psi$  as

$$(3.14) \quad \tilde{\psi}(x) = \min\{\psi(x), \psi_-(x') + \psi(x'')\} = \psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x''),$$

where  $(x', x'')$  ranges over all pairs in the non-primitive part of the coproduct  $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ , and  $\psi_-$  is assumed defined by induction on all the lower degree terms  $x'$  in the Hopf algebra as

$$(3.15) \quad \psi_-(x) := T(\tilde{\psi}(x)) = T(\min\{\psi(x), \psi_-(x') + \psi(x'')\}) = T\left(\psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x'')\right).$$

By the  $\oplus$ -linearity of  $T$ , this is the same as

$$\psi_-(x) = \min\{T(\psi(x)), T(\psi_-(x') + \psi(x''))\} = T(\psi(x)) \oplus \bigoplus T(\psi_-(x') \odot \psi(x'')).$$

The positive part of the factorization is then obtained as the convolution product

$$(3.16) \quad \psi_+(x) := (\psi_- \star \psi)(x) = \min\{\psi_-(x), \psi(x), \psi_-(x') + \psi(x'')\} = \min\{\psi_-(x), \tilde{\psi}(x)\} = \psi_-(x) \oplus \tilde{\psi}(x).$$

We need to check that  $\psi_{\pm}$  satisfy (3.10). We have  $\psi_-(xy) = T \min\{\psi(x) + \psi(y), \psi_-(xy)' + \psi((xy)'')\}$ , where we can decompose the terms  $(xy)'$  and  $(xy)''$  in terms of  $x, y, x'$  and  $x'', y'$  and

$y''$ . This gives

$$(3.17) \quad \psi_-(xy) = T \min \left\{ \begin{array}{l} \psi(x) + \psi(y), \\ \psi_-(x) + \psi(y), \\ \psi_-(y) + \psi(x), \\ \psi_-(y') + \psi(xy''), \\ \psi_-(x') + \psi(x''y), \\ \psi_-(xy') + \psi(y''), \\ \psi_-(x'y) + \psi(x''), \\ \psi_-(x'y') + \psi(x''y'') \end{array} \right\}.$$

Using associativity and commutativity of  $\oplus$  and  $\oplus$ -additivity of  $T$ , we can group these terms together into

$$\psi_-(xy) = \min\{\alpha(x, y, x', y'), \beta(x, y, x', y')\},$$

where we have

$$(3.18) \quad \alpha(x, y, x', y') = T \min\{\psi_-(x) + \psi(y), \psi(x) + \psi_-(y), \psi_-(xy') + \psi(y''), \psi_-(x'y) + \psi(x'')\}$$

$$(3.19) \quad \beta(x, y, x', y') = T \min\{\psi(x) + \psi(y), \psi_-(y') + \psi(xy''), \psi_-(x') + \psi(x''y), \psi_-(x'y') + \psi(x''y'')\}$$

Assuming inductively that

$$\psi_-(uv) = \psi_-(u) + \psi_-(v),$$

for all terms  $u$  and  $v$  in  $\mathcal{H}$  of degrees  $\deg(u) + \deg(v) < \deg(xy)$ , and using the fact that  $T$  is  $\oplus$ -additive, we can rewrite the term  $\alpha(x, y, x', y')$  of (3.18) as

$$(3.20) \quad \begin{aligned} \alpha(x, y, x', y') &= T \min\{\psi_-(x) + \tilde{\psi}(y), \tilde{\psi}(x) + \psi_-(y)\} \\ &= \min\{T(T(\tilde{\psi}(x)) + \tilde{\psi}(y)), T(\tilde{\psi}(x) + T(\tilde{\psi}(y)))\} \end{aligned}$$

and we can write the term  $\beta(x, y, x', y')$  of (3.19) as

$$(3.21) \quad \beta(x, y, x', y') = T \min\{\tilde{\psi}(x) + \tilde{\psi}(y)\} = \min\{T(\tilde{\psi}(x) + \tilde{\psi}(y))\}.$$

Thus, we have

$$(3.22) \quad \begin{aligned} \psi_-(xy) &= \min\{T(\tilde{\psi}(x) + \tilde{\psi}(y)), T(T(\tilde{\psi}(x)) + \tilde{\psi}(y)), T(\tilde{\psi}(x) + T(\tilde{\psi}(y)))\} \\ &= T(\tilde{\psi}(x) \odot \tilde{\psi}(y)) \oplus T(T(\tilde{\psi}(x)) \odot \tilde{\psi}(y)) \oplus T(\tilde{\psi}(x) \odot T(\tilde{\psi}(y))). \end{aligned}$$

Since the operator  $T$  satisfies the Rota–Baxter identity (3.12) with  $\lambda = 1$ , we can rewrite the above as

$$\psi_-(xy) = T(\tilde{\psi}(x)) \odot T(\tilde{\psi}(y)) = T(\tilde{\psi}(x)) + T(\tilde{\psi}(y)) = \psi_-(x) + \psi_-(y).$$

The fact that  $\psi_+(xy) = \psi_+(x) + \psi_+(y)$  then follows from  $\psi_+ = \psi_- \star \psi$ .  $\square$

#### 4. THERMODYNAMIC ROTA–BAXTER STRUCTURES AND BIRKHOFF FACTORIZATIONS

In Theorem 3.9 we have used the associativity and commutativity properties of the tropical addition  $\oplus$ , in reordering the terms in  $\psi_-(xy)$  to prove it satisfies  $\psi_-(xy) = \psi_-(x) + \psi_-(y)$ . Thus, in extending the result to thermodynamic semirings, we will focus on the case of thermodynamic deformations  $\oplus_{\beta, S}$ , where  $S$  is the Shannon entropy, since in this case both associativity and commutativity continue to hold for the deformed addition  $\oplus_{\beta, S}$ .

**Definition 4.1.** Let  $\mathbb{S}_{\beta, S}$  be thermodynamic deformations of a semiring  $\mathbb{S}$ , with operations  $\oplus_{\beta, S}$  and  $\odot$ , and with  $S$  the Shannon entropy. An operator  $T : \mathbb{S}_{\beta, S} \rightarrow \mathbb{S}_{\beta, S}$  is  $\oplus_{\beta, S}$ -linear if, for all  $f_1, f_2 \in \mathbb{S}_{\beta, S}$  and all  $\alpha, \gamma \in \mathbb{T}$ ,

$$(4.1) \quad T(\alpha \odot f_1 \oplus_{\beta, S} \gamma \odot f_2) = \alpha \odot T(f_1) \oplus_{\beta, S} \gamma \odot T(f_2).$$

**4.1. Classical and thermodynamic Rota–Baxter operators.** As in the case of min-plus semirings  $\mathbb{S} = C(X, \mathbb{T})$  with pointwise  $\oplus$  and  $\odot$  operations, we can similarly define Rota–Baxter structures on their thermodynamic deformations  $\mathbb{S}_{\beta,S}$ . In the case of weight  $\lambda > 0$  we have the following.

**Definition 4.2.** A thermodynamic semiring  $\mathbb{S}_{\beta,S}$  is a Rota–Baxter semiring of weight  $\lambda > 0$  if there is a  $\oplus_{\beta,S}$ -additive map  $T : \mathbb{S}_{\beta,S} \rightarrow \mathbb{S}_{\beta,S}$ , which for all  $f_1, f_2 \in \mathbb{S}_{\beta,S}$  satisfies the identity

$$(4.2) \quad T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2)) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log \lambda.$$

The case with  $\lambda < 0$  is analogous: we have the following.

**Definition 4.3.** A thermodynamic semiring  $\mathbb{S}_{\beta,S}$  is a Rota–Baxter semiring of weight  $\lambda < 0$  if there is a  $\oplus_{\beta,S}$ -additive map  $T : \mathbb{S}_{\beta,S} \rightarrow \mathbb{S}_{\beta,S}$ , which for all  $f_1, f_2 \in \mathbb{S}_{\beta,S}$  satisfies the identity

$$(4.3) \quad T(f_1) \odot T(f_2) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2)).$$

**Theorem 4.4.** Let  $\mathcal{R}$  be a commutative ring and  $\mathbb{S}$  a min-plus semiring, logarithmically related as in Definition 3.1. Given  $T : \mathbb{S} \rightarrow \mathbb{S}$ , define a new map  $\mathcal{T} : \mathcal{R} \rightarrow \mathcal{R}$  by setting

$$\mathcal{T}(e^{-\beta f}) := e^{-\beta T(f)},$$

for  $a = e^{-\beta f}$  in  $\text{Dom}(\log) \subset \mathcal{R}$ . Then  $T$  satisfies the Rota–Baxter identity (4.2) or (4.3) of weight  $\lambda$  if and only if  $\mathcal{T}$  satisfies the ordinary Rota–Baxter identity

$$\mathcal{T}(e^{-\beta f_1})\mathcal{T}(e^{-\beta f_2}) = \mathcal{T}(\mathcal{T}(e^{-\beta f_1})e^{-\beta f_2}) + \mathcal{T}(e^{-\beta f_1}\mathcal{T}(e^{-\beta f_2})) + \lambda_{\beta} \mathcal{T}(e^{-\beta f_1}e^{-\beta f_2}).$$

of weight  $\lambda_{\beta} = \lambda^{-\beta}$ , for  $\lambda > 0$ , or weight  $\lambda_{\beta} = -|\lambda|^{-\beta}$  for  $\lambda < 0$ .

*Proof.* In the case  $\lambda > 0$ , we write the left-hand-side of the Rota–Baxter identity of weight  $\lambda_{\beta}$  for  $\mathcal{T}$  as

$$\mathcal{T}(e^{-\beta f_1})\mathcal{T}(e^{-\beta f_2}) = e^{-\beta(T(f_1)+T(f_2))}$$

while the right-hand-side gives

$$\begin{aligned} & \mathcal{T}(\mathcal{T}(e^{-\beta f_1})e^{-\beta f_2}) + \mathcal{T}(e^{-\beta f_1}\mathcal{T}(e^{-\beta f_2})) + \lambda_{\beta} \mathcal{T}(e^{-\beta(f_1+f_2)}) \\ &= \mathcal{T}(e^{-\beta(T(f_1)+f_2)}) + \mathcal{T}(e^{-\beta(f_1+T(f_2))}) + \lambda_{\beta} e^{-\beta T(f_1+f_2)} \\ &= e^{-\beta T(T(f_1)+f_2)} + e^{-\beta T(f_1+T(f_2))} + e^{-\beta(T(f_1+f_2)-\beta^{-1} \log \lambda_{\beta})}. \end{aligned}$$

This gives the identity

$$T(f_1) + T(f_2) = -\beta^{-1} \log(e^{-\beta T(T(f_1)+f_2)} + e^{-\beta T(f_1+T(f_2))} + e^{-\beta(T(f_1+f_2)-\beta^{-1} \log \lambda_{\beta})}).$$

For  $\lambda_{\beta} = \lambda^{-\beta}$ , this is equivalently written as

$$T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2)) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log \lambda.$$

In the case with  $\lambda < 0$ , we write the left-hand-side of the Rota–Baxter identity for  $\mathcal{T}$  as

$$\mathcal{T}(e^{-\beta f_1})\mathcal{T}(e^{-\beta f_2}) - \lambda_{\beta} \mathcal{T}(e^{-\beta f_1}e^{-\beta f_2}) = e^{-\beta(T(f_1)+T(f_2))} + e^{-\beta(T(f_1+f_2)-\beta^{-1} \log(-\lambda_{\beta}))}$$

and the right-hand-side

$$\mathcal{T}(e^{-\beta(T(f_1)+f_2)}) + \mathcal{T}(e^{-\beta(f_1+T(f_2))}) = e^{-\beta T(T(f_1)+f_2)} + e^{-\beta T(f_1+T(f_2))}.$$

This gives the identity

$$-\beta^{-1} \log(e^{-\beta(T(f_1)+T(f_2))} + e^{-\beta(T(f_1+f_2)-\beta^{-1} \log(-\lambda_{\beta}))}) = -\beta^{-1} \log(e^{-\beta T(T(f_1)+f_2)} + e^{-\beta T(f_1+T(f_2))}).$$

For  $\lambda_{\beta} = -|\lambda|^{-\beta}$ , this is equivalently written as

$$T(f_1) \odot T(f_2) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2)).$$

□

We also check that linearity (in the ordinary sense) for the operator  $\mathcal{T}$  corresponds to  $\oplus_{\beta,S}$ -linearity for  $T$ . For a semiring  $\mathbb{S} = C(X, \mathbb{T})$ , we extend  $\mathcal{T}$  to  $\mathcal{R} = C(X, \mathbb{R})$  by requiring that  $\mathcal{T}(-e^{-\beta f}) := -\mathcal{T}(e^{-\beta f})$ .

**Proposition 4.5.** *Let  $\mathcal{T}(e^{-\beta f}) := e^{-\beta T(f)}$ , as in Theorem 4.4. Then the operator  $\mathcal{T}$  is  $\mathbb{R}$ -linear if and only if the operator  $T$  is  $\oplus_{\beta,S}$ -linear.*

*Proof.* We have

$$\begin{aligned} \mathcal{T}(e^{-\beta f_1} + e^{-\beta f_2}) &= \mathcal{T}(e^{-\beta(-\beta^{-1} \log(e^{-\beta f_1} + e^{-\beta f_2}))}) \\ &= \mathcal{T}(e^{-\beta(f_1 \oplus_{\beta,S} f_2)}) = e^{-\beta(T(f_1 \oplus_{\beta,S} f_2))}. \end{aligned}$$

We also have

$$\begin{aligned} \mathcal{T}(e^{-\beta f_1}) + \mathcal{T}(e^{-\beta f_2}) &= e^{-\beta T(f_1)} + e^{-\beta T(f_2)} \\ &= e^{-\beta(-\beta^{-1} \log(e^{-\beta T(f_1)} + e^{-\beta T(f_2)}))} = e^{-\beta T(f_1) \oplus_{\beta,S} T(f_2)}, \end{aligned}$$

hence  $\mathcal{T}(e^{-\beta f_1} + e^{-\beta f_2}) = \mathcal{T}(e^{-\beta f_1}) + \mathcal{T}(e^{-\beta f_2})$  if and only if  $T(f_1 \oplus_{\beta,S} f_2) = T(f_1) \oplus_{\beta,S} T(f_2)$ . Moreover, for  $\alpha \in \mathbb{R}_+^*$ , we have

$$\mathcal{T}(\alpha e^{-\beta f}) = \mathcal{T}(e^{-\beta(f - \beta^{-1} \log \alpha)}) = e^{-\beta T(f - \beta^{-1} \log \alpha)}.$$

This agrees with

$$\alpha \mathcal{T}(e^{-\beta f}) = \alpha e^{-\beta T(f)} = e^{-\beta(T(f) - \beta^{-1} \log \alpha)}$$

if and only if, for all  $f \in C(X, \mathbb{R})$  and all  $\alpha \in \mathbb{R}_+^*$  we have  $T(f - \beta^{-1} \log \alpha) = T(f) - \beta^{-1} \log \alpha$ . The two properties  $T(f_1 \oplus_{\beta,S} f_2) = T(f_1) \oplus_{\beta,S} T(f_2)$  and  $T(f + \lambda) = T(f) + \lambda$ , for all  $f, f_1, f_2 \in C(X, \mathbb{R})$  and all  $\lambda \in \mathbb{R}$ , are equivalent to  $\oplus_{\beta,S}$ -linearity (4.1).  $\square$

**4.2. Birkhoff factorization in thermodynamic semirings.** Let  $\mathcal{H}$  be a graded connected commutative Hopf algebra and  $\psi : \mathcal{H} \rightarrow \mathbb{S}_{\beta,S}$  satisfying  $\psi(xy) = \psi(x) + \psi(y)$ .

**Definition 4.6.** *Let  $T : \mathbb{S}_{\beta,S} \rightarrow \mathbb{S}_{\beta,S}$  be a Rota–Baxter operator of weight  $\lambda = +1$ , as in Definition 4.2. The Bogolyubov–Parashchuk preparation of  $\psi$  is defined as*

$$(4.4) \quad \tilde{\psi}_{\beta,S}(x) = \psi(x) \oplus_{\beta,S} \bigoplus_{\beta,S} \psi_-(x') + \psi(x'') = -\beta^{-1} \log \left( e^{-\beta \psi(x)} + \sum e^{-\beta(\psi_-(x') + \psi(x''))} \right),$$

where  $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ , and where  $\psi_-(x) = T\tilde{\psi}(x)$ .

**Remark 4.7.** When  $\beta \rightarrow \infty$ , the Bogolyubov–Parashchuk preparation  $\tilde{\psi}_{\beta,S}(x)$  converges to the preparation (3.14).

**Lemma 4.8.** *Given  $\psi : \mathcal{H} \rightarrow \mathbb{S}_{\beta,S}$  satisfying  $\psi(xy) = \psi(x) + \psi(y)$ , for all  $x, y \in \mathcal{H}$ , let  $\phi_\beta(x) := e^{-\beta \psi(x)}$ . Then  $\phi_\beta(xy) = \phi_\beta(x) \phi_\beta(y)$ , for all  $x, y \in \mathcal{H}$ . The Bogolyubov–Parashchuk preparation of  $\psi$  satisfies  $\tilde{\phi}_\beta(x) = e^{-\beta \tilde{\psi}(x)}$ , for all  $x \in \mathcal{H}$ , where*

$$\tilde{\phi}_\beta(x) := \phi_\beta(x) + \sum \mathcal{T}(\tilde{\phi}_\beta(x')) \phi_\beta(x''),$$

with  $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ , and where the operator  $\mathcal{T}$  is defined by  $\mathcal{T}(e^{-\beta f}) := e^{-\beta T(f)}$  and  $\mathcal{T}(-e^{-\beta f}) := -\mathcal{T}(e^{-\beta f})$ .

*Proof.* The multiplicativity of  $\phi_\beta$  is evident. For the Bogolyubov–Parashchuk preparation, we inductively assume that for the lower degree terms  $\tilde{\phi}_\beta(x') = e^{-\beta \tilde{\psi}(x')}$ . Then the result follows from the relation between the operators  $T$  and  $\mathcal{T}$ .  $\square$

**Definition 4.9.** Given  $\psi_1$  and  $\psi_2$  from  $\mathcal{H}$  to  $\mathbb{S}_{\beta,S}$  with  $\psi_i(xy) = \psi_i(x) + \psi_i(y)$ , we set

$$(4.5) \quad (\psi_1 \star_{\beta} \psi_2)(x) = \bigoplus_{\beta,S} (\psi_1(x^{(1)}) + \psi_2(x^{(2)})),$$

where  $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$ .

**Lemma 4.10.** Let  $\phi_{i,\beta}(x) = e^{-\beta\psi_i(x)}$ . Then  $(\phi_{1,\beta} \star \phi_{2,\beta})(x) = e^{-\beta(\psi_1 \star_{\beta} \psi_2)(x)}$ .

*Proof.* The usual product of Hopf algebra characters is given by

$$(\phi_{1,\beta} \star \phi_{2,\beta})(x) = \sum \phi_{1,\beta}(x^{(1)}) \phi_{2,\beta}(x^{(2)}),$$

where  $\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$ . This can be written equivalently as

$$\begin{aligned} \sum e^{-\beta(\psi_{1,\beta}(x^{(1)}) + \phi_{2,\beta}(x^{(2)}))} &= e^{-\beta(\beta^{-1} \log(\sum e^{-\beta(\psi_{1,\beta}(x^{(1)}) + \phi_{2,\beta}(x^{(2)}))})} \\ &= e^{-\beta \bigoplus_{\beta,S} (\psi_1(x^{(1)}) + \psi_2(x^{(2)}))} = e^{-\beta(\psi_1 \star_{\beta} \psi_2)(x)}. \end{aligned}$$

□

The construction of Birkhoff factorizations in thermodynamic semirings is then given by the following.

**Theorem 4.11.** Let  $T : \mathbb{S}_{\beta,S} \rightarrow \mathbb{S}_{\beta,S}$  be a  $\oplus_{\beta,S}$ -additive Rota–Baxter operator of weight  $\lambda = +1$ , in the sense of Definition 4.2. Then there is a factorization  $\psi_{\beta,+} = \psi_{\beta,-} \star_{\beta} \psi$ , where  $\psi_{\beta,\pm}$  are defined as

$$(4.6) \quad \psi_{\beta,-}(x) = T(\tilde{\psi}_{\beta}(x)) = -\beta^{-1} \log \left( e^{-\beta T(\psi(x))} + \sum e^{-\beta T(\psi_{-}(x') + \psi(x''))} \right)$$

$$(4.7) \quad \psi_{\beta,+}(x) = -\beta^{-1} \log \left( e^{-\beta \psi_{\beta,-}(x)} + e^{-\beta \tilde{\psi}_{\beta}(x)} \right).$$

The positive and negative parts of the Birkhoff factorization satisfy  $\psi_{\beta,\pm}(xy) = \psi_{\beta,\pm}(x) + \psi_{\beta,\pm}(y)$ .

*Proof.* By Lemma 4.8, the statement is analogous to showing the existence of a factorization  $\phi_{\beta,+} = \phi_{\beta,-} \star_{\beta} \phi$  for  $\phi_{\beta}(x) = e^{-\beta\psi(x)}$ , with  $\phi_{\beta,-}(x) = e^{-\beta\psi_{\beta,-}(x)}$  and  $\phi_{\beta,+}(x) = e^{-\beta\psi_{\beta,+}(x)}$ , and such that  $\phi_{\beta,\pm}(xy) = \phi_{\beta,\pm}(x)\phi_{\beta,\pm}(y)$ . Such a factorization can be constructed inductively by setting

$$\phi_{\beta,-}(x) = \phi_{\beta}(x) + \sum \phi_{\beta,-}(x') \phi_{\beta}(x''),$$

$$\phi_{\beta,+}(x) = \phi_{\beta,-}(x) + \tilde{\phi}_{\beta}(x),$$

where, according to Lemma 4.8 and Propositions 4.4 and 4.5,

$$\phi_{\beta,-}(x) = \mathcal{T}(\tilde{\phi}_{\beta}(x)) = e^{-\beta T(\tilde{\psi}_{\beta}(x))} = e^{-\beta \psi_{\beta,-}(x)}.$$

The multiplicative property for  $\phi_{\beta,+}$  follows from that of  $\phi_{\beta,-}$  and of  $\phi_{\beta}$ . Thus, it suffices to show  $\phi_{\beta,-}(xy) = \phi_{\beta,-}(x)\phi_{\beta,-}(y)$ . We proceed as in the case of the usual Birkhoff factorization, and identify the terms in

$$\phi_{\beta,-}(xy) = \mathcal{T}(\phi_{\beta}(xy) + \sum \phi_{\beta,-}((xy)') \phi_{\beta}((xy)''))$$

with

$$\mathcal{T}(\tilde{\phi}_{\beta}(x)\tilde{\phi}_{\beta}(y)) + \mathcal{T}(\mathcal{T}(\tilde{\phi}_{\beta}(x))\tilde{\phi}_{\beta}(y)) + \mathcal{T}(\tilde{\phi}_{\beta}(x)\mathcal{T}(\tilde{\phi}_{\beta}(y))).$$

Using Proposition 4.4 and the resulting Rota–Baxter identity of weight  $\lambda = +1$  for  $\mathcal{T}$ , we identify this with

$$\mathcal{T}(\tilde{\phi}_{\beta}(x))\mathcal{T}(\tilde{\phi}_{\beta}(y)) = \phi_{\beta,-}(x)\phi_{\beta,-}(y).$$

□



**Remark 4.12.** In the limit when  $\beta \rightarrow \infty$ , the Birkhoff factorization of Theorem 4.11 converges to the Birkhoff factorization of Theorem 3.9.

## 5. VON NEUMANN ENTROPY AND ROTA–BAXTER STRUCTURES

We now consider again the case of matrices. Recall from Theorem 1.2.8 of [13] that if  $\mathcal{R}$  is a commutative  $\mathbb{R}$ -algebra, endowed with a Rota–Baxter operator  $\mathcal{T}$  of weight  $\lambda$ , then  $\mathcal{T}$  induces a Rota–Baxter operator (which we still denote  $\mathcal{T}$ ), of the same weight, on the ring of matrices  $M_n(\mathcal{R})$ , by applying  $\mathcal{T}$  coordinate-wise,  $\mathcal{T}(A) = (\mathcal{T}(a_{ij}))$ , for  $A = (a_{ij})$ .

**Proposition 5.1.** *Let  $\mathcal{R}$  be a commutative  $\mathbb{R}$ -algebra and  $\mathbb{S}$  be a min-plus semiring, related by the property that, for  $A \in M_n(\mathbb{S})$ , the matrix  $e^{-\beta A} \in M_n(\mathcal{R})$ . Let  $\mathcal{D} \subset M_n(\mathcal{R})$  denote the set of matrices of the form  $e^{-\beta A}$ , for  $A \in M_n(\mathbb{S})$ . Let  $\mathcal{T}$  be a Rota–Baxter operator of weight  $+1$  on  $\mathcal{R}$ , and let  $(M_n(\mathcal{R}), \mathcal{T})$  be Rota–Baxter structure described above. Then setting  $\mathcal{T}(e^{-\beta A}) = e^{-\beta \mathcal{T}(A)}$  defines an operator  $T : M_n(\mathbb{S}) \rightarrow M_n(\mathbb{S})$  that satisfies the following type of Rota–Baxter identity*

$$\text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(A) \boxplus T(B)) = \text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(T(A) \boxplus B)) \oplus_{\beta, S} \text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(A \boxplus T(B))) \oplus_{\beta, S} \text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(A) \boxplus T(B)),$$

where  $\mathcal{N}$  is the von Neumann entropy,  $S$  is the Shannon entropy, and  $\boxplus$  denotes the direct sum of matrices.

*Proof.* The Rota–Baxter identity for  $\mathcal{T}$  on  $\mathcal{R}$  gives

$$\begin{aligned} \mathcal{T}(\text{Tr}(e^{-\beta A}))\mathcal{T}(\text{Tr}(e^{-\beta B})) &= \mathcal{T}(\mathcal{T}(\text{Tr}(e^{-\beta A}))\text{Tr}(e^{-\beta B})) + \mathcal{T}(\text{Tr}(e^{-\beta A})\mathcal{T}(\text{Tr}(e^{-\beta B}))) \\ &\quad + \mathcal{T}(\text{Tr}(e^{-\beta A})\text{Tr}(e^{-\beta B})). \end{aligned}$$

Notice that the induced Rota–Baxter structure on  $M_n(\mathcal{R})$  satisfies  $\mathcal{T}(\text{Tr}(A)) = \text{Tr}(\mathcal{T}(A))$ . Thus, using this fact together with  $\mathcal{T}(e^{-\beta A}) = e^{-\beta \mathcal{T}(A)}$ , we can rewrite the above as

$$\begin{aligned} \text{Tr}(e^{-\beta \mathcal{T}(A)})\text{Tr}(e^{-\beta \mathcal{T}(B)}) &= \mathcal{T}(\text{Tr}(e^{-\beta \mathcal{T}(A)})\text{Tr}(e^{-\beta \mathcal{T}(B)})) + \mathcal{T}(\text{Tr}(e^{-\beta A})\text{Tr}(e^{-\beta \mathcal{T}(B)})) \\ &\quad + \mathcal{T}(\text{Tr}(e^{-\beta A})\text{Tr}(e^{-\beta B})). \end{aligned}$$

We can then identify the products of traces with the trace of the tensor product of matrices, which gives

$$\text{Tr}(e^{-\beta \mathcal{T}(A)} \otimes e^{-\beta \mathcal{T}(B)}) = \mathcal{T}(\text{Tr}(e^{-\beta \mathcal{T}(A)} \otimes e^{-\beta B})) + \mathcal{T}(\text{Tr}(e^{-\beta A} \otimes e^{-\beta \mathcal{T}(B)})) + \mathcal{T}(\text{Tr}(e^{-\beta A} \otimes e^{-\beta B})).$$

Moreover, for matrix exponentials,  $\exp(A) \otimes \exp(B) = \exp(A \boxplus B)$ , where here  $\boxplus$  is the direct sum of matrices. Thus, we obtain

$$\begin{aligned} \text{Tr}(e^{-\beta(T(A) \boxplus T(B))}) &= \text{Tr}(e^{-\beta(T(T(A) \boxplus B))}) + \text{Tr}(e^{-\beta(T(A \boxplus T(B))}) \\ &\quad + \text{Tr}(e^{-\beta(T(A) \boxplus T(B))}). \end{aligned}$$

This then gives

$$\begin{aligned} -\beta^{-1} \log \text{Tr}(e^{-\beta(T(A) \boxplus T(B))}) &= -\beta^{-1} \log(\text{Tr}(e^{-\beta(T(T(A) \boxplus B))}) + \text{Tr}(e^{-\beta(T(A \boxplus T(B))}) \\ &\quad + \text{Tr}(e^{-\beta(T(A) \boxplus T(B))})), \end{aligned}$$

or equivalently

$$\text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(A) \boxplus T(B)) = -\beta^{-1} \log \left( e^{-\beta \text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(T(A) \boxplus B))} + e^{-\beta \text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(A \boxplus T(B))} + e^{-\beta \text{Tr}_{\beta, \mathcal{N}}^{\oplus}(T(A) \boxplus T(B))} \right),$$

hence (5.1) follows.  $\square$

## 6. ROTA–BAXTER STRUCTURES OF WEIGHT ONE: THERMODYNAMICS AND WITT RINGS

We analyze here some examples for Rota–Baxter operators of weight +1 on thermodynamic semirings, derived from classical examples of weight-one Rota–Baxter algebras. We also show that the same examples of weight-one Rota–Baxter algebras can be used to induce Rota–Baxter structures on Witt rings. We interpret the effect of the resulting Rota–Baxter operators applied to zeta functions of varieties, regarded as elements of Witt rings, as in [23].

**6.1. Partial sums.** Consider the  $\mathbb{R}$ -algebra  $\mathcal{R}$  of  $\mathbb{R}$ -valued sequences  $a = (a_1, a_2, a_3, \dots) = (a_n)_{n=1}^\infty$ , with coordinate-wise addition and multiplication, and let  $\mathcal{T} : \mathcal{R} \rightarrow \mathcal{R}$  be the linear operator that maps the sequence  $(a_1, a_2, a_3, \dots, a_n, \dots)$  to  $(0, a_1, a_1 + a_2, \dots, \sum_{k=1}^{n-1} a_k, \dots)$ . The operator  $\mathcal{T}$  is a Rota–Baxter operator of weight +1, see Example 1.1.6 of [13].

**Lemma 6.1.** *The Rota–Baxter algebra  $(\mathcal{R}, \mathcal{T})$  of weight +1 described above determines a Rota–Baxter structure of weight +1 on the thermodynamic semi-rings  $\mathbb{S}_{\beta, S}$  of functions  $f : \mathbb{N} \rightarrow \mathbb{T} = \mathbb{R} \cup \{\infty\}$ , with the pointwise operations  $\oplus_{\beta, S}$  and  $\odot$ , with Rota–Baxter operator*

$$(6.1) \quad (Tf)(n) = \bigoplus_{\beta, S, k=1, \dots, n-1} f(k),$$

for  $n \geq 2$  and  $(Tf)(1) = \infty$ .

*Proof.* For  $\mathcal{R}$  as above, let  $\mathcal{D} \subset \mathcal{R}$  be the subset of sequences with values in  $\mathbb{R}_+$ , which we can write as  $a_n = e^{-\beta c_n}$ , when  $a_n > 0$  and zero otherwise. We have  $(\mathcal{T}a)_1 = 0$  and  $(\mathcal{T}a)_n = \sum_{k=1}^{n-1} a_k$  for  $n \geq 2$ . Define  $(Tc)_n = -\beta^{-1} \log(\mathcal{T}a)_n$ , so that

$$(Tc)_n = \begin{cases} \infty & n = 1 \\ -\beta^{-1} \log \left( \sum_{k=1}^{n-1} e^{-\beta c_k} \right) & n \geq 2. \end{cases}$$

□

**6.2.  $q$ -integral.** Let  $\mathcal{R} = \mathbb{R}[[t]]$  be the ring of formal power series with real coefficients. Let  $\mathcal{T}$  be the linear operator  $(\mathcal{T}\alpha)(t) = \sum_{k=1}^\infty \alpha(q^k t)$ , for  $q$  not a root of unity. The operator  $\mathcal{T}$  is a Rota–Baxter operator of weight +1. The operator  $\mathcal{T}$  maps a single power  $t^n$  to  $q^n t^n / (1 - q^n)$ , hence it restricts to a Rota–Baxter operator of weight +1 on the subring of polynomials  $\mathbb{R}[t]$ , see Example 1.1.8 of [13].

**Lemma 6.2.** *Let  $\mathbb{S}$  be the thermodynamic semiring of formal power series  $\mathbb{S}_{\beta, S} = \mathbb{R}[[t]] \cup \{\infty\}$  with the operations  $(\gamma_1 \oplus_{\beta, S} \gamma_2)(t) = -\beta^{-1} \log(e^{-\beta \gamma_1(t)} + e^{-\beta \gamma_2(t)})$  and with  $(\gamma_1 \odot \gamma_2)(t) = \gamma_1(t) + \gamma_2(t)$ . Then the Rota–Baxter algebra of weight +1, given by the data  $(\mathcal{R}, \mathcal{T})$  described above, induces a Rota–Baxter structure of weight +1 on  $\mathbb{S}_{\beta, S}$  by*

$$(6.2) \quad (T\gamma)(t) = \bigoplus_{\beta, S, k=1}^\infty \gamma(q^k t).$$

*Proof.* Let  $\mathcal{D} \subset \mathcal{R}$  be the subset of formal series with  $a_0 = 1$ , that is,  $\mathcal{D} = 1 + t\mathbb{R}[[t]]$ . Then for  $\alpha \in \mathcal{D}$  and  $\gamma(t) = \log \alpha(t)$ , we define an operator  $T$  by the relation  $\mathcal{T}(e^{-\beta \gamma(t)}) = e^{-\beta (T\gamma)(t)}$ . This gives  $e^{-\beta (T\gamma)(t)} = \sum_{k=1}^\infty e^{-\beta \gamma(q^k t)}$ , that is,

$$(T\gamma)(t) = -\beta^{-1} \log \left( \sum_{k=1}^\infty e^{-\beta \gamma(q^k t)} \right) = \bigoplus_{\beta, S, k=1}^\infty \gamma(q^k t).$$

□

**6.3. Rota–Baxter structures on Witt rings.** For a commutative ring  $R$ , the Witt ring  $W(R)$  can be identified with the set of formal power series with  $a_0 = 1$ , that is, the set  $1 + tR[[t]]$  with the Witt addition given by the usual product of formal power series and the Witt multiplication  $\star$  uniquely determined by the rule

$$(1 - at)^{-1} \star (1 - bt)^{-1} = (1 - abt)^{-1},$$

for  $a, b \in R$ . There is an injective ring homomorphism  $g : W(R) \rightarrow R^{\mathbb{N}}$ ,  $g(\alpha) = (\alpha_1, \alpha_2, \dots, \alpha_r, \dots)$ , where the addition and multiplication operations on  $R^{\mathbb{N}}$  are component-wise. The sequence  $g_n(\alpha) = \alpha_n$  is known as the “ghost coordinates” of  $\alpha$ . Upon writing elements of the Witt ring  $W(R)$  in the exponential form

$$\exp \left( \sum_{r \geq 1} \alpha_r \frac{t^r}{r} \right),$$

one sees that the ghost coordinates are the coefficients of

$$t \frac{1}{\alpha} \frac{d\alpha}{dt} = \sum_{r \geq 1} \alpha_r t^r.$$

**Lemma 6.3.** *A linear operator  $\mathcal{T} : R^{\mathbb{N}} \rightarrow R^{\mathbb{N}}$  is a Rota–Baxter operator of weight  $\lambda$  on  $R^{\mathbb{N}}$  if and only if the operator  $\mathcal{T}_W$  defined on the Witt ring  $W(R)$  so that, when taking ghost components  $g(\mathcal{T}_W(\alpha)) = \mathcal{T}(g(\alpha))$  is a Rota–Baxter operator of weight  $\lambda$  on  $W(R)$ .*

*Proof.* the Rota–Baxter identity for  $\mathcal{T}_W$  is of the form

$$\mathcal{T}_W(\alpha_1) \star \mathcal{T}_W(\alpha_2) = \mathcal{T}_W(\alpha_1 \star \mathcal{T}_W(\alpha_2)) +_W \mathcal{T}_W(\mathcal{T}_W(\alpha_1) \star \alpha_2) +_W \lambda \star \mathcal{T}_W(\alpha_1 \star \alpha_2))$$

with  $+_W$  the sum in  $W(R)$ . When taking ghost components, this gives

$$g(\mathcal{T}_W(\alpha_1) \star \mathcal{T}_W(\alpha_2)) = g(\mathcal{T}_W(\alpha_1 \star \mathcal{T}_W(\alpha_2))) + g(\mathcal{T}_W(\mathcal{T}_W(\alpha_1) \star \alpha_2)) + \lambda g(\mathcal{T}_W(\alpha_1 \star \alpha_2))$$

which gives the Rota–Baxter identity for  $\mathcal{T}$ ,

$$\mathcal{T}(g(\alpha_1))\mathcal{T}(g(\alpha_2)) = \mathcal{T}(g(\alpha_1)\mathcal{T}(g(\alpha_2))) + \mathcal{T}(\mathcal{T}(g(\alpha_1))g(\alpha_2)) + \lambda \mathcal{T}(g(\alpha_1)g(\alpha_2)).$$

The injectivity of the ghost map shows we can run the implication backward.  $\square$

In addition to the Witt product  $\star$  of the Witt ring  $W(R)$ , which corresponds to the coordinate-wise product of the ghost components, one can introduce a convolution product on  $W(R)$ , which is induced by the power-series product of the ghost maps.

**Definition 6.4.** *For  $\alpha, \gamma \in W(R)$ , with  $\alpha = \exp(\sum_{r \geq 1} \alpha_r t^r / r)$  and  $\gamma = \exp(\sum_{r \geq 1} \gamma_r t^r / r)$ , the convolution product is given as*

$$(6.3) \quad \alpha \otimes \gamma := \exp \left( \sum_{n \geq 1} \left( \sum_{r+\ell=n} \alpha_r \gamma_\ell \right) \frac{t^n}{n} \right).$$

Notice that  $\alpha \otimes \gamma$  is defined so that the ghost  $g(\alpha \otimes \gamma) = \sum_{n \geq 1} \sum_{r+\ell=n} \alpha_r \gamma_\ell t^n$  is the product as power series  $g(\alpha) \bullet g(\gamma)$  of the ghosts  $g(\alpha) = \sum_{r \geq 1} \alpha_r t^r$  and  $g(\gamma) = \sum_{r \geq 1} \gamma_r t^r$ .

**Lemma 6.5.** *A linear operator  $\mathcal{T} : R[[t]] \rightarrow R[[t]]$  is a Rota–Baxter operator of weight  $\lambda$  if and only if the operator  $\mathcal{T}_W : W(R) \rightarrow W(R)$  defined by  $g(\mathcal{T}_W(\alpha)) = \mathcal{T}(g(\alpha))$  satisfies the Rota–Baxter identity of weight  $\lambda$  with respect to the convolution product (6.3),*

$$(6.4) \quad \mathcal{T}_W(\alpha_1) \otimes \mathcal{T}_W(\alpha_2) = \mathcal{T}_W(\alpha_1 \otimes \mathcal{T}_W(\alpha_2)) +_W \mathcal{T}_W(\mathcal{T}_W(\alpha_1) \otimes \alpha_2) +_W \lambda \mathcal{T}_W(\alpha_1 \otimes \alpha_2),$$

where  $+_W$  is the addition in  $W(R)$ .

*Proof.* After composing with the ghost map, (6.4) gives

$$\mathcal{T}(g(\alpha_1)) \bullet \mathcal{T}(g(\alpha_2)) = \mathcal{T}(g(\alpha_1) \bullet \mathcal{T}(g(\alpha_2))) + \mathcal{T}(\mathcal{T}(g(\alpha_1)) \bullet g(\alpha_2)) + \lambda \mathcal{T}(g(\alpha_1) \bullet g(\alpha_2)),$$

where  $\bullet$  denotes the product as formal power series. This is the Rota–Baxter identity for  $\mathcal{T}$  on  $R[[t]]$ . The injectivity of the ghost map shows the two conditions are equivalent.  $\square$

We consider then the example of Rota–Baxter operator of weight one given by partial sums.

**Proposition 6.6.** *Let  $\mathcal{R} = R^{\mathbb{N}}$  with the Rota–Baxter operator of weight +1 given by*

$$\mathcal{T} : (a_1, a_2, \dots, a_n, \dots) \mapsto (0, a_1, a_1 + a_2, \dots, \sum_{k=1}^{n-1} a_k, \dots).$$

*The resulting Rota–Baxter operator  $\mathcal{T}_W$  of weight +1 on the Witt ring  $W(R)$  is given by convolution product with the multiplicative unit (for the usual Witt product)  $\mathbb{I} = (1 - t)^{-1}$  of  $W(R)$ ,*

$$(6.5) \quad \mathcal{T}_W(\alpha) = \alpha \circledast \mathbb{I}.$$

*Proof.* According to Lemma 6.3, the Rota–Baxter operator  $\mathcal{T}_W$  on  $W(R)$  is given by

$$\mathcal{T}_W(\alpha) = \exp \left( \sum_{n \geq 2} \sum_{k=1}^{n-1} \alpha_k \frac{t^n}{n} \right),$$

for  $\alpha = \exp(\sum_{n \geq 1} \alpha_n t^n / n)$ . The ghost of  $\alpha$  is given by  $g(\alpha) = \sum_{n \geq 1} \alpha_n t^n$  and we can identify the series

$$\sum_{n \geq 2} \sum_{k=1}^{n-1} \alpha_k t^n = g(\alpha) \bullet \frac{t}{1-t},$$

where  $\bullet$  denotes the product of formal power series. The identity

$$-t \frac{d}{dt} \log(1-t) = \frac{t}{1-t}$$

then shows that we can identify the above with the product of power series  $g(\alpha) \bullet g(\mathbb{I})$ , hence by construction  $\mathcal{T}_W(\alpha) = \alpha \circledast \mathbb{I}$ .  $\square$

We also consider the example of the weight-one Rota–Baxter operator on power series given by the  $q$ -integral.

**Proposition 6.7.** *Let  $\mathcal{R} = R[[t]]$  with the Rota–Baxter operator  $\mathcal{T}_q$  of weight +1 given by the  $q$ -integral (where  $q \in R$  is not a root of unity). Then the operator  $\mathcal{T}_{W,q}$  on  $W(R)$  defined by  $g(\mathcal{T}_{W,q}(\alpha)) = \mathcal{T}_q(g(\alpha))$  is a Rota–Baxter operator of weight one with respect to the convolution product (6.3). It is explicitly given by  $\mathcal{T}_W(\alpha)(t) = \prod_{k \geq 1} \alpha(q^k t)$ .*

*Proof.* The operator  $\mathcal{T}_{W,q}$  acts as

$$\mathcal{T}_{W,q}(\exp(\sum_{r \geq 1} \alpha_r \frac{t^r}{r})) = \exp(\sum_{r \geq 1} \sum_{k \geq 1} \alpha_r \frac{q^{kr} t^r}{r}) = \prod_{k \geq 1} \exp(\sum_{r \geq 1} \alpha_r \frac{(q^k t)^r}{r}).$$

Notice that the product  $\prod_k \alpha(q^k t)$ , which is the product as power series, is the *addition* in the Witt ring  $W(R)$ , so the operator  $\mathcal{T}_W$  has the same form as the  $q$ -integral operator  $\mathcal{T}$ , simply replacing the sum in  $R[[t]]$  with the sum in  $W(R)$ . By Lemma 6.5,  $\mathcal{T}_{W,q}$  satisfies the identity

$$\mathcal{T}_{W,q}(\alpha_1) \circledast \mathcal{T}_{W,q}(\alpha_2) = \mathcal{T}_{W,q}(\alpha_1 \circledast \mathcal{T}_{W,q}(\alpha_2)) +_W \mathcal{T}_{W,q}(\mathcal{T}_{W,q}(\alpha_1) \circledast \alpha_2) +_W \mathcal{T}_{W,q}(\alpha_1 \circledast \alpha_2).$$

$\square$

**6.4. Applications to zeta functions.** For varieties (or schemes) over finite fields, the Hasse–Weil zeta function is given by

$$Z(X, t) = \exp \left( \sum_{r \geq 1} \#X(\mathbb{F}_{q^r}) \frac{t^r}{r} \right).$$

Equivalently, it can be written as

$$Z(X, t) = \prod_{r \geq 1} (1 - t^r)^{-a_r(X)} = \prod_{x \in X_{cl}} (1 - t^{\deg(x)})^{-1},$$

where  $a_r(X) = \#\{x \in X_{cl} \mid [k(x) : \mathbb{F}_q] = r\}$  and  $X_{cl}$  is the set of closed points of  $X$ . The zeta function satisfies the properties

$$Z(X \sqcup Y, t) = Z(X, t)Z(Y, t),$$

for a disjoint union  $X \sqcup Y$  and

$$Z(X \times Y, t) = Z(X, t) \star Z(Y, t),$$

where  $\star$  is the product in the Witt ring. Thus, it is natural to consider zeta functions of varieties as elements of a Witt ring, [23]. In particular, this means that we can apply the Rota–Baxter operators on Witt rings described above to zeta functions of varieties.

**Corollary 6.8.** *Let  $\mathcal{T}_W$  be the Rota–Baxter operator of Proposition 6.6. For  $X$  a variety (or scheme) over  $\mathbb{F}_q$ ,*

$$\mathcal{T}_W(Z(X, t)) = Z(X, t) \otimes Z(\text{Spec}(\mathbb{F}_q), t).$$

*Proof.* This is immediate from Proposition 6.6, since  $Z(\text{Spec}(\mathbb{F}_q), t) = \exp(\sum_{r \geq 1} \frac{t^r}{r}) = (1-t)^{-1}$ .  $\square$

Recall that the Grothendieck ring of varieties (or schemes of finite type) over  $\mathbb{F}_q$  is generated by isomorphism classes  $[X]$  with the inclusion-exclusion relation  $[X] = [Y] + [X \setminus Y]$  for closed  $Y \subset X$  and the product  $[X \times Y] = [X][Y]$ . The zeta function  $Z(X, t) = Z([X], t)$  factors as a ring homomorphism from the Grothendieck ring to the Witt ring. In the Grothendieck ring, the Lefschetz motive is the class of the affine line  $\mathbb{L} = [\mathbb{A}^1]$ . In the theory of motives it is customary to localize the Grothendieck ring by inverting the Lefschetz motive. The Tate motive is the formal inverse  $\mathbb{L}^{-1}$ . For the Rota–Baxter structure of Proposition 6.7 we then have the following.

**Corollary 6.9.** *Let  $X$  be a variety (or scheme) over  $k = \mathbb{F}_q$  and  $[X]$  its Grothendieck class. Consider the Rota–Baxter structure of Proposition 6.7 with Rota–Baxter operator  $\mathcal{T}_{W,q}$  or  $\mathcal{T}_{W,q^{-1}}$ . These give*

$$\mathcal{T}_{W,q}(Z(X, t)) = \prod_{k \geq 1} Z([X] \mathbb{L}^k, t), \quad \mathcal{T}_{W,q^{-1}}(Z(X, t)) = \prod_{k \geq 1} Z([X] \mathbb{L}^{-k}, t),$$

where  $\mathbb{L}$  is the Lefschetz motive and  $\mathbb{L}^{-1}$  is the Tate motive.

*Proof.* In the case of the Lefschetz motive we have  $Z(X, q^k t) = Z(X \times \mathbb{A}^k, t) = Z([X] \mathbb{L}^k, t)$ . In the case of the Tate motive, we do not have the geometric space replacing  $X \times \mathbb{A}^k$ , but the property that the zeta function is a ring homomorphism from the Grothendieck ring to the Witt ring gives  $Z(X, q^{-k} t) = Z([X] \mathbb{L}^{-k}, t)$ .  $\square$

**Corollary 6.10.** *Let  $\mathcal{T}_{W,q}$  and  $\mathcal{T}_{W,q^{-1}}$  be as above. Then the operators  $\tilde{\mathcal{T}}_{W,q^{\pm 1}} := -_{wid} -_W \mathcal{T}_{W,q^{\pm 1}}$  are also Rota–Baxter operators of weight +1. For  $X$  a variety over  $\mathbb{F}_q$ , they give*

$$\tilde{\mathcal{T}}_{W,q^{\pm 1}}(Z(X, t)) = \prod_{k \geq 0} Z([X] \mathbb{L}^{\pm k}, t)^{-1}.$$

*Proof.* It is a simple general fact that, if  $\mathcal{T}$  is a Rota–Baxter operator of weight  $+1$  then  $\tilde{\mathcal{T}} = -id - \mathcal{T}$  is also a Rota–Baxter operator of weight  $+1$ . Thus, the operators  $\tilde{\mathcal{T}}_{W,q\pm 1}$  satisfy the identity (6.4) with  $\lambda = +1$ . The explicit expression for  $\tilde{\mathcal{T}}_{W,q\pm 1}(Z(X,t))$  then follows exactly as in Corollary 6.9.  $\square$

## 7. ROTA–BAXTER OPERATORS OF WEIGHT $-1$

We have seen in the previous sections how to construct Birkhoff factorizations in min-plus semirings and their thermodynamical deformations, based on the use of Rota–Baxter operators of weight  $\lambda = +1$ .

In the usual setting of Birkhoff factorizations in perturbative quantum field theory, [7], [10], [11], one constructs the Birkhoff factorization using a Rota–Baxter operator  $\mathcal{T}$  of weight  $\lambda = -1$  by setting  $\phi_-(x) = -\mathcal{T}(\tilde{\phi}(x))$ , where  $-\mathcal{T}$  is a Rota–Baxter operator of weight  $+1$ .

In the semiring setting, one cannot proceed in the same way. However, it is still possible to construct Birkhoff factorizations from Rota–Baxter operator of weight  $\lambda = -1$ , under some additional conditions on the operator.

Let  $\mathbb{S} = C(X, \mathbb{T})$  with the pointwise min-plus operations  $\oplus$  and  $\odot$ .

**Proposition 7.1.** *Let  $\psi : \mathcal{H} \rightarrow \mathbb{S}$  be a min-plus character, and let  $T : \mathbb{S} \rightarrow \mathbb{S}$  be a Rota–Baxter operator of weight  $-1$ , in the sense of Definition 3.8. Then there is a Birkhoff factorization  $\psi_+ = \psi_- \star \psi$ . If, moreover, the Rota–Baxter operator  $T$  satisfies  $T(f_1 + f_2) \geq T(f_1) + T(f_2)$ , then  $\psi_-$  and  $\psi_+$  are also min-plus characters.*

*Proof.* As in the case of Theorem 3.9, we define the two sides of the factorization as  $\psi_-(x) := T(\tilde{\psi}(x))$  and  $\psi_+(x) := (\psi_- \star \psi)(x) = \min\{\psi_-(x), \tilde{\psi}(x)\}$ , where the preparation  $\tilde{\psi}(x)$  is defined as in (3.14). To show that  $\psi_-(xy) = \psi_-(x) + \psi_-(y)$ , we again list the terms  $(xy)'$  and  $(xy)''$  as in Theorem 3.9 and obtain

$$\psi_-(xy) = \min\{T(\tilde{\psi}(x) + \tilde{\psi}(y)), T(T(\tilde{\psi}(x)) + \tilde{\psi}(y)), T(\tilde{\psi}(x) + T(\tilde{\psi}(y)))\}.$$

The Rota–Baxter identity of weight  $-1$  for the operator  $T$  then gives

$$\psi_-(xy) = \min\{T(\tilde{\psi}(x) + \tilde{\psi}(y)), T(\tilde{\psi}(x)) + T(\tilde{\psi}(y))\}.$$

If the operator  $T$  satisfies  $T(f_1 + f_2) \geq T(f_1) + T(f_2)$ , for all  $a, b \in \mathbb{S}$ , we then have

$$\psi_-(xy) = T(\tilde{\psi}(x) + \tilde{\psi}(y)) = \psi_-(x) + \psi_-(y).$$

$\square$

The following observations show that there are choices of semiring Rota–Baxter operators satisfying  $T(a + b) \geq T(a) + T(b)$ .

**Proposition 7.2.** *For  $X$  a compact Hausdorff space, let  $\mathbb{S} = C(X, \mathbb{R})$ , with the pointwise  $\oplus, \odot$  operations. Let  $T : \mathbb{S} \rightarrow \mathbb{S}$  be an idempotent linear operator (in the ordinary sense) on the underlying algebra  $C(X, \mathbb{R})$  with respect to the usual additive structure on  $C(X, \mathbb{R})$ . Then  $T$  is (trivially) a semiring Rota–Baxter operator satisfying the hypothesis of Theorem 3.9.*

*Proof.* Since  $T$  is linear, it satisfies  $T(a + b) = T(a) + T(b)$  and the Rota–Baxter relation simply becomes  $T(a + b) = \min\{T^2(a) + T(b), T(a) + T^2(b)\}$ , which is certainly satisfied if  $T$  is idempotent,  $T^2 = T$ , since the right-hand-side is then also equal to  $T(a) + T(b)$ .  $\square$

**Example 7.3.** Let  $\mathbb{S} = C(X, \mathbb{T})$ , as in Proposition 7.2, where  $X$  is a totally disconnected compact Hausdorff space (a Cantor set). Then for any clopen subset  $Y \subset X$  the operator  $T = T_Y : \mathbb{S} \rightarrow \mathbb{S}$  given by ordinary multiplication by the characteristic function of  $Y$ ,  $T_Y : f(x) \mapsto \chi_Y(x)f(x)$  is a semiring Rota–Baxter operator satisfying the conditions of Theorem 3.9.

**Remark 7.4.** Examples of semiring Rota–Baxter operators satisfying the conditions of Theorem 3.9, but not arising from linear operators in the usual sense, can be constructed using idempotent superadditive operators. These occur, for instance, in potential theory: we refer the reader to §10 of [2] for some relevant constructions.

**7.1. Other forms of Birkhoff factorization of weight  $-1$  in min-plus semirings.** We consider here a different possible way of obtaining Birkhoff factorization for min-plus semirings, using Rota–Baxter operators of weight  $-1$ . This method involves the use of two related Rota–Baxter operators, generalizing the roles of the operators  $\mathcal{T}$  and  $1 - \mathcal{T}$  in the original commutative algebra case. As in the case of Proposition 7.1, we need superadditivity conditions on these operators to obtain that the parts of the factorization are still min-plus character.

**Definition 7.5.** Let  $(\mathbb{S}, \oplus, \odot)$  be a min-plus semiring. Let  $T : \mathbb{S} \rightarrow \mathbb{S}$  and  $\tilde{T} : \mathbb{S} \rightarrow \mathbb{S}$  be  $\oplus$ -additive Rota–Baxter operators of weight  $-1$  (as in Definition 3.8) satisfying the relations

$$(7.1) \quad T\alpha = \alpha \oplus \tilde{T}\alpha, \quad \forall \alpha \in \mathbb{S}$$

$$(7.2) \quad \tilde{T}(\alpha \odot \beta) \oplus \tilde{T}(\alpha) \odot \tilde{T}(\beta) = \tilde{T}(T(\alpha) \odot \beta \oplus \alpha \odot T(\beta)), \quad \forall \alpha, \beta \in \mathbb{S}.$$

Given a min-plus character  $\psi : \mathcal{H} \rightarrow \mathbb{S}$ , a  $(T, \tilde{T})$ -Birkhoff factorization of  $\psi$  is given by the pair

$$(7.3) \quad \psi_-(x) = T\tilde{\psi}(x) = T(\psi(x) \oplus \bigoplus_{(x', x'')} \psi_-(x') \odot \psi(x''))$$

$$(7.4) \quad \psi_+(x) = \tilde{T}\tilde{\psi}(x) = \tilde{T}(\psi(x) \oplus \bigoplus_{(x', x'')} \psi_-(x') \odot \psi(x'')),$$

where the  $\oplus$ -sums are over pairs  $(x', x'')$  in the non-primitive part of the coproduct of  $\mathcal{H}$ ,  $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$ .

**Proposition 7.6.** Let the data  $(\mathbb{S}, \oplus, \odot)$ , with  $T : \mathbb{S} \rightarrow \mathbb{S}$  and  $\tilde{T} : \mathbb{S} \rightarrow \mathbb{S}$ , be as in Definition 7.5. If the operators  $T$  and  $\tilde{T}$  are superadditive then the resulting pieces  $\psi_{\pm}$  of the factorization are min-plus characters. Moreover, the terms  $\psi_{\pm}$  of the  $(T, \tilde{T})$ -Birkhoff factorization are related by

$$(7.5) \quad \psi_-(x) = \min\{\tilde{\psi}(x), \psi_+(x)\} = \min\{\psi_+ \star \psi(x), \tilde{\psi}(x') + \psi(x'')\}.$$

*Proof.* We first need to check that  $\psi_{\pm}(xy) = \psi_{\pm}(x) + \psi_{\pm}(y)$ . The case of  $\psi_-$  is proved as in Theorem 3.9. In the case of  $\psi_+$ , by proceeding as in Theorem 3.9, we see that

$$\psi_+(xy) = \tilde{T} \min\{T(\tilde{\psi}(x)) + \tilde{\psi}(y), \tilde{\psi}(x) + T(\tilde{\psi}(y)), \tilde{\psi}(x) + \tilde{\psi}(y)\}.$$

Using the  $\oplus$ -additivity (monotonicity) of  $\tilde{T}$  and (7.2) we write the above as

$$\psi_+(xy) = \min\{\tilde{T}(\tilde{\psi}(x)) + \tilde{T}(\tilde{\psi}(y)), \tilde{T}(\tilde{\psi}(x) + \tilde{\psi}(y))\}.$$

If  $\tilde{T}$  is subadditive, the minimum is equal to

$$\psi_+(xy) = \tilde{T}(\tilde{\psi}(x)) + \tilde{T}(\tilde{\psi}(y)) = \psi_+(x) + \psi_+(y).$$

Thus, both sides of the factorization satisfy  $\psi_{\pm}(xy) = \psi_{\pm}(x) + \psi_{\pm}(y)$ . We have

$$\psi_-(x) = T(\tilde{\psi}(x)) = \min\{T(\psi(x)), T(T(\tilde{\psi}(x')) + \psi(x''))\}.$$



The identity (7.1) implies that we have

$$\psi_-(x) = \min\{\tilde{\psi}(x), \psi_+(x)\}$$

which we write also as  $\min\{\psi(x), \psi_+(x), \psi_-(x') + \psi(x'')\}$ . We then use (7.1) and rewrite  $\psi_-(x') = \min\{\tilde{\psi}(x'), \tilde{T}(\tilde{\psi}(x'))\}$ . Thus we obtain  $\psi_-(x) = \min\{\psi(x), \psi_+(x), \psi_+(x') + \psi(x''), \tilde{\psi}(x') + \psi(x'')\}$ , where  $\min\{\psi(x), \psi_+(x), \psi_+(x') + \psi(x'')\}$  is the convolution product  $(\psi_+ \star \psi)(x)$ , hence the statement follows.  $\square$

## 8. MIN-PLUS CHARACTERS AND THERMODYNAMICS

We consider here some examples of min-plus characters  $\psi : \mathcal{H} \rightarrow \mathbb{S}$ , satisfying  $\psi(xy) = \psi(x) + \psi(y)$ . We focus on the case where  $\mathcal{H}$  a Hopf algebra of graphs, namely the commutative algebra generated by connected finite graphs with coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma.$$

**8.1. Inclusion–exclusion functions on graphs.** We consider real valued functions  $\tau$  on a set of graphs, that satisfy an inclusion-exclusion property. Namely, if  $\Gamma = \Gamma_1 \cup \Gamma_2$  with intersection  $\gamma = \Gamma_1 \cap \Gamma_2$ , then

$$(8.1) \quad \tau(\Gamma) = \tau(\Gamma_1) + \tau(\Gamma_2) - \tau(\gamma).$$

Examples of such functions can be constructed by assigning a “cost function” to the sets of vertices and edges of a graph. Let  $F_E = \{f_e : e \in E(\Gamma)\}$  and  $F_V = \{f_v : v \in V(\Gamma)\}$ . Then setting  $\tau(\Gamma) = \sum_{v \in V(\Gamma)} f_v + \sum_{e \in E(\Gamma)} f_e$  gives a function that satisfies inclusion-exclusion. (One of the sums may be trivial if one only assigns vertex or edge labels.)

In particular, for a disjoint union  $\Gamma = \Gamma_1 \sqcup \Gamma_2$  we have  $\tau(\Gamma) = \tau(\Gamma_1) + \tau(\Gamma_2)$ , hence we can view such a function  $\tau$  as a morphism  $\tau : \mathcal{H} \rightarrow \mathbb{T}$ , where  $\mathcal{H}$  is the Hopf algebra of graphs and  $\mathbb{T}$  is the tropical semiring, satisfying  $\tau(xy) = \tau(x) + \tau(y)$ , hence it defines a min-plus character. The function  $\tau$  obtained as above may depend on a set of parameters, so that the  $f_e$  and  $f_v$  are functions of these parameters, so that we can think of  $\tau : \mathcal{H} \rightarrow \mathbb{S}$  as a min-plus character to some min-plus semiring of functions.

**8.2. Examples from computation.** Following §4.6 of [15], we consider a Hopf algebra of “flow charts” for computation, namely graphs endowed with acyclic orientations, so that the flow through the graph, from the input vertices to the output vertices, represents the structure of a computation. Vertices are decorated by elementary operations on partial recursive functions and edges are decorated by partial recursive functions that are inputs and outputs of the vertex operations, see [15], [16], and see also the discussion in [9] on generalizations of Manin’s Hopf algebra of flow charts. The computation associated to a graph  $\Gamma$  depends on a set of parameters.

We consider min-plus characters  $\psi : \mathcal{H} \rightarrow \mathbb{S}$ , where the choice of the target min-plus semiring  $\mathbb{S}$  accounts for the dependence on parameters. Typical such characters would be the running time of the computation (if computations associated to different connected components of the graph are run sequentially) or the memory size involved in the computation, with  $\psi(\Gamma) = \psi(\Gamma_1) + \psi(\Gamma_2) = \psi(\Gamma_1) \odot \psi(\Gamma_2)$ , for a disjoint union  $\Gamma = \Gamma_1 \sqcup \Gamma_2$ .

In the theory of computation, one approach to characterize the complexity of computable functions in a machine-independent way is by considering a sequence of machines in a given class (1-tape machines, multiple tape machines, etc.) and associate to each machine in the sequence a step-counting function, which is the number of steps of tape (or computing time) that the machine

takes to compute a given recursive function (or infinity if the computation does not stop). This method is the basis for speed-up and compression theorems, see [5] for more details.

Suppose given a decorated graph  $\Gamma \in \mathcal{H}$  with decorations by recursive functions and operations as in Manin's Hopf algebra of flow charts. Then, given a class of machines, we let  $\psi_n(\Gamma)$  be the step-counting function of the  $n$ -th machine in the class, when it computes the output of  $\Gamma$ . We set  $\psi_n(\Gamma) = \infty$  if the  $n$ -th machine does not halt when fed the input of  $\Gamma$ . We also assume that, if  $\Gamma$  has several components, the computations are done sequentially, so that  $\psi_n(\Gamma_1 \sqcup \Gamma_2) = \psi_n(\Gamma_1) + \psi_n(\Gamma_2)$ , hence all the  $\psi_n$  are  $\mathbb{T}$ -valued min-plus characters. Moreover, in the case of a union that is not disjoint, one can assume that the step-counting functions  $\psi_n$  satisfy an inclusion-exclusion principle  $\psi_n(\Gamma_1 \cup \Gamma_2) = \psi_n(\Gamma_1) + \psi_n(\Gamma_2) - \psi_n(\Gamma_1 \cap \Gamma_2)$ .

We then consider the Rota–Baxter operator of weight +1 given by the partial sum, as in §6.1. The preparation of the character  $\psi(\Gamma) = (\psi_n(\Gamma))_{n \in \mathbb{N}}$  is given by

$$\tilde{\psi}_n(\Gamma) = \min\{\psi_n(\Gamma), \psi_n(\Gamma/\gamma) + \sum_{k=1}^{n-1} \tilde{\psi}_k(\gamma)\},$$

where the minimum is taken over subgraphs  $\gamma$ . Notice that, by the inclusion-exclusion property, we are comparing the size (number of steps/computing time) of  $\psi_n(\Gamma) = \psi_n(\Gamma/\gamma) + \psi_n(\gamma) - \psi_n(\partial\gamma)$ , with the size of  $\psi_n(\Gamma/\gamma) + \sum_{k=1}^{n-1} \psi_k(\gamma)$ ; and then the minimum of these with the further terms  $\sum_{k=1}^{n-1} (\psi_k(\gamma') + \psi_k(\gamma/\gamma'))$ , for subgraphs  $\gamma' \subset \gamma$ , and so on, in the recursive structure of the  $\tilde{\psi}_k(\gamma)$ . At each step, one identifies smaller graphs inside  $\Gamma$  for which either the cumulative computational time of all the previous machines in the series is small, or the additional computational cost of the “interior” part of the subgraph  $\gamma \setminus \partial\gamma$  is small.

In the case of a graph  $\Gamma$  for which the  $n$ -th machine does not halt, so  $\psi_n(\Gamma) = \infty$ , the character  $\tilde{\psi}_n$  can be finite, provided the following conditions are realized:

- The source of the infinite computational time for the  $n$ -th machine was localized in an area  $\gamma \setminus \partial\gamma$  of the graph  $\Gamma$ , that is,  $\psi_n(\Gamma/\gamma) < \infty$ .
- None of the previous machines had infinite computational time on this region of the graph:  $\psi_k(\gamma) < \infty$  for all  $k = 1, \dots, n-1$ .

**8.3. Nearest neighbor potentials and Markov random fields.** Given a subgraph  $\gamma \subset \Gamma$  we denote by  $\partial\gamma$  the subgraph with  $E(\partial\gamma)$  the set of edges in  $E(\Gamma)$  with  $\partial e$  consisting of a vertex in  $V(\gamma)$  and a vertex in  $V(\Gamma) \setminus V(\gamma)$ . The set of vertices  $V(\partial\gamma)$  is the union of these endpoints, for all  $e \in E(\partial\gamma)$ . In particular, for a vertex  $v \in V(\Gamma)$  we write  $\partial(v)$  for the set of vertices  $V(\partial\{v\}) \subset V(\Gamma)$ .

A *Markov random field* on a graph is a map  $\pi : \mathcal{P}(V(\Gamma)) \rightarrow \mathbb{R}$ , where  $\mathcal{P}(V(\Gamma))$  is the set of subsets of  $V(\Gamma)$ , satisfying  $\pi(A) > 0$  for all  $A \in \mathcal{P}(V(\Gamma))$  and

$$(8.2) \quad \frac{\pi(A \cup \{v\})}{\pi(A)} = \frac{\pi(A \cap \partial(v)) \cup \{v\})}{\pi(A \cap \partial(v))},$$

for all  $A \in \mathcal{P}(V(\Gamma))$  and all  $v \in V(\Gamma)$ , see §1 of [22].

A *nearest neighbor potential* on a graph is a function  $\mathcal{W} : \mathcal{P}(V(\Gamma)) \rightarrow \mathbb{R}$  satisfying

$$(8.3) \quad \mathcal{W}(A \cup \{v\}) - \mathcal{W}(A) = \mathcal{W}(A \cap \partial(v)) \cup \{v\} - \mathcal{W}(A \cap \partial(v)),$$

for all  $A \in \mathcal{P}(V(\Gamma))$  and all  $v \in V(\Gamma)$ , see §1 of [22]. Unlike [22], here we do not require normalizations for  $\pi$  by  $\pi(\emptyset)$ , or of  $\mathcal{W}$ , by the partition function  $Z = \sum_A \exp(\mathcal{W}(A))$ .

We extend the notion of nearest neighbor potentials and Markov random fields from a single graph to a (finite or infinite) family of graphs.

**Definition 8.1.** Given a family  $\mathcal{G}$  of finite graphs a Markov random field on  $\mathcal{G}$  is a function  $\pi : \mathcal{G} \rightarrow \mathbb{R}$  satisfying  $\pi(\Gamma) > 0$  for all  $\Gamma \in \mathcal{G}$  and

$$(8.4) \quad \frac{\pi(\Gamma \cup \{v\})}{\pi(\Gamma)} = \frac{\pi(\Gamma \cap \partial(v)) \cup \{v\}}{\pi(\Gamma \cap \partial(v))},$$

whenever the graphs  $\Gamma \cup \{v\}$ ,  $\Gamma \cap \partial(v)$  and  $(\Gamma \cap \partial(v)) \cup \{v\}$  belong to  $\mathcal{G}$ . A nearest neighbor potential on  $\mathcal{G}$  is a function  $\mathcal{W} : \mathcal{G} \rightarrow \mathbb{R}$  satisfying

$$(8.5) \quad \mathcal{W}(\Gamma \cup \{v\}) - \mathcal{W}(\Gamma) = \mathcal{W}((\Gamma \cap \partial(v)) \cup \{v\}) - \mathcal{W}(\Gamma \cap \partial(v)),$$

whenever  $\Gamma \cup \{v\}$ ,  $\Gamma \cap \partial(v)$  and  $(\Gamma \cap \partial(v)) \cup \{v\}$  belong to  $\mathcal{G}$ .

We recover the usual notion of [22] if we fix a graph  $\Gamma$  and we define  $\mathcal{G}$  to be the set of all induced subgraphs of  $\Gamma$ , namely all subgraphs determined by a choice of a subset  $A$  of vertices of  $\Gamma$ , and all the edges of  $\Gamma$  between those vertices.

**Lemma 8.2.** If  $\mathcal{W} : \mathcal{G} \rightarrow \mathbb{R}$  is a nearest neighbor potential on  $\mathcal{G}$ , then, for all  $\beta > 0$ , setting  $\pi_\beta(\Gamma) = e^{-\beta \mathcal{W}(\Gamma)}$  defines a random Markov field  $\pi_\beta : \mathcal{G} \rightarrow \mathbb{R}$ .

*Proof.* This follows immediately by adapting the general observation of §1 of [22], that if  $\pi$  is a Markov random field then  $\mathcal{W}(A) = \log(\pi(A))$  is a nearest neighbor potential and, conversely, given a nearest neighbor potential  $\mathcal{W}$ , setting  $\pi(A) = \exp(\mathcal{W}(A))$  gives a Markov random field.  $\square$

Let  $\mathcal{G}$  be a family of finite graphs, closed under disjoint unions, and  $\mathcal{H} = \mathcal{H}(\mathcal{G})$  the Hopf algebra generated as a commutative algebra by the connected components of elements of  $\mathcal{G}$  with coproduct  $\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum \gamma \otimes \Gamma/\gamma$  where the non-primitive part of the coproduct is the sum over all pairs of a subgraph  $\gamma$  and the quotient graph  $\Gamma/\gamma$  (where every component of  $\gamma$  is contracted to a vertex) such that both  $\gamma$  and  $\Gamma/\gamma$  belong to  $\mathcal{G}$ .

**Lemma 8.3.** For  $\mathcal{G}$  and  $\mathcal{H} = \mathcal{H}(\mathcal{G})$  as above, a nearest-neighbor potential  $\mathcal{W} : \mathcal{G} \rightarrow \mathbb{R}$  defines a min-plus character  $\mathcal{W} : \mathcal{H} \rightarrow \mathbb{T}$ .

*Proof.* It is immediate to check that (8.5) implies  $\mathcal{W}(\Gamma) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2)$  for a disjoint union  $\Gamma = \Gamma_1 \sqcup \Gamma_2$ . In fact, by inductively adding vertices of  $\Gamma_2$  we have  $\mathcal{W}(\Gamma_1 \cup \{v\}) - \mathcal{W}(\Gamma_1) = \mathcal{W}(\{v\})$  for  $v \in V(\Gamma_2)$  and assuming that for  $\gamma \subset \Gamma_2$  with  $\#V(\gamma) = n$  we have  $\mathcal{W}(\Gamma_1 \cup \gamma) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\gamma)$  we obtain  $\mathcal{W}(\Gamma_1 \cup \gamma \cup \{v\}) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\gamma) + \mathcal{W}((\gamma \cap \partial(v)) \cup \{v\}) - \mathcal{W}(\gamma \cap \partial(v)) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\gamma) + \mathcal{W}(\gamma \cup \{v\}) - \mathcal{W}(\gamma) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\gamma \cup \{v\})$ , for all  $v \in V(\Gamma_2) \setminus V(\gamma)$ .  $\square$

Similarly, for more general min-plus semirings  $\mathbb{S}$ , min-plus characters  $\psi : \mathcal{H} \rightarrow \mathbb{S}$  that depend only on the vertex set of graphs define  $\mathbb{S}$ -valued nearest neighbor potentials.

**Lemma 8.4.** Let  $\mathbb{S}$  be a min-plus semiring, and let  $\psi : \mathcal{H} \rightarrow \mathbb{S}$  be a min-plus character with the property that the value  $\psi(\Gamma) \in \mathbb{S}$  depends only on the set  $V(\Gamma)$  of vertices of  $\Gamma$ . Then  $\psi$  is a  $\mathbb{S}$ -valued nearest neighbor potential.

*Proof.* We have  $\psi(\Gamma) = \psi((\Gamma \cap \partial(v)) \cup (\Gamma \setminus \partial(v)))$ . If the value of  $\psi$  only depends on the vertex set, then the latter is equal to  $\psi((\Gamma \cap \partial(v)) \sqcup (\Gamma \setminus \partial(v)))$ . Since  $\psi$  is a min-plus character, this is  $\psi(\Gamma \cap \partial(v)) + \psi(\Gamma \setminus \partial(v))$ . Thus, we have  $\psi(\Gamma \cup \{v\}) - \psi(\Gamma) = \psi((\Gamma \cap \partial(v)) \cup \{v\}) + \psi(\Gamma \setminus \partial(v)) - \psi(\Gamma \cap \partial(v)) - \psi(\Gamma \setminus \partial(v)) = \psi((\Gamma \cap \partial(v)) \cup \{v\}) - \psi(\Gamma \setminus \partial(v))$ .  $\square$

We can then view the Birkhoff factorization of min-plus characters in thermodynamic semirings as a method for generating new Markov random fields from given ones.

**Proposition 8.5.** *Let  $\mathcal{W} : \mathcal{H} \rightarrow \mathbb{S}$  be a nearest neighbor potential, with associated Markov random field  $\pi_\beta(\Gamma) = e^{-\beta\mathcal{W}(\Gamma)}$ . Let  $\mathbb{S}_{\beta,S}$  be the thermodynamic deformation of  $\mathbb{S}$  with  $S$  the Shannon entropy, and let  $T : \mathbb{S}_{\beta,S} \rightarrow \mathbb{S}_{\beta,S}$  be an  $\otimes_{\beta,S}$ -linear weight-one Rota–Baxter operator. Let  $\mathcal{W}_{\beta,\pm} : \mathcal{H} \rightarrow \mathbb{S}_{\beta,S}$  be the two parts of the Birkhoff factorization of  $\mathcal{W}$ . Then  $\pi_{\beta,\pm}(\Gamma) = e^{-\beta\mathcal{W}_\pm(\Gamma)}$  are Markov random fields.*

*Proof.* The factorization is given by

$$\begin{aligned}\mathcal{W}_{\beta,-}(\Gamma) &= -\beta^{-1} \log \left( e^{-\beta T\mathcal{W}(\Gamma)} + \sum e^{-\beta T(\mathcal{W}_{\beta,-}(\gamma) + \mathcal{W}(\Gamma/\gamma))} \right), \\ \mathcal{W}_{\beta,+}(\Gamma) &= -\beta^{-1} \log \left( e^{-\beta\mathcal{W}_{\beta,-}(\Gamma)} + e^{-\beta\tilde{\mathcal{W}}_\beta(\Gamma)} \right),\end{aligned}$$

where

$$\tilde{\mathcal{W}}_\beta(\Gamma) = -\beta^{-1} \log \left( e^{-\beta\mathcal{W}(\Gamma)} + \sum e^{-\beta(\mathcal{W}_{\beta,-}(\gamma) + \mathcal{W}(\Gamma/\gamma))} \right).$$

According to Theorem 4.11,  $\mathcal{W}_{\beta,\pm} : \mathcal{H} \rightarrow \mathbb{S}_{\beta,S}$  are min-plus characters. Moreover, the explicit expression above shows that, if  $\mathcal{W}(\Gamma)$  depends only on the set  $V(\Gamma)$  of vertices of  $\Gamma$ , then so do also the  $\mathcal{W}_{\beta,\pm}(\Gamma)$ . Thus, by Lemma 8.4 the  $\mathcal{W}_{\beta,\pm}$  are  $\mathbb{S}_{\beta,S}$ -valued nearest neighbor potentials, and  $\pi_{\beta,\pm}$  are Markov random fields.  $\square$

**8.4. Algebro-geometric Feynman rules and polynomial countability.** In perturbative quantum field theory, one can write the Feynman integrals in the parametric form as (unrenormalized) period integrals on the complement of the (affine) graph hypersurface  $X_\Gamma \subset \mathbb{A}^{\#E(\Gamma)}$ , defined by the vanishing of the graph polynomial  $\Psi_\Gamma(t) = \sum_T \prod_{e \notin E(T)} t_e$ , where the sum is over spanning trees of the Feynman graph  $\Gamma$  and  $t = (t_e)_{e \in E(\Gamma)} \in \mathbb{A}^{\#E(\Gamma)}$ , see [18] for a general overview.

It was observed in [1] that the class in the Grothendieck ring of the affine hypersurface complement  $Y_\Gamma := \mathbb{A}^{\#E(\Gamma)} \setminus X_\Gamma$  determines a morphism of commutative rings, from the Hopf algebra of Feynman graphs to the Grothendieck ring, since it satisfies

$$(8.6) \quad [Y_\Gamma] = [Y_{\Gamma_1}] \cdot [Y_{\Gamma_2}]$$

when  $\Gamma$  is a disjoint union  $\Gamma = \Gamma_1 \sqcup \Gamma_2$ . Such morphisms were termed “algebro-geometric Feynman rules” in [1], where examples based on Chern classes of singular varieties were also constructed, with values in a suitable Grothendieck group of immersed conical varieties.

Recall that a variety  $X$  defines over  $\mathbb{Z}$  is polynomially countable if for all the mod  $p$  reductions  $X_p$ , the counting functions of points over  $\mathbb{F}_q$ , with  $q = p^r$ , is a polynomial in  $q$  with  $\mathbb{Z}$ -coefficients, namely  $N(X, q) := \#X_p(\mathbb{F}_q) = P_X(q)$ . Polynomial countability is a consequence (and, modulo certain conjectures on motives, equivalent) to the class in the Grothendieck ring  $[X] = P_X(\mathbb{L})$  being in the polynomial subring  $\mathbb{Z}[\mathbb{L}]$  generated by the Lefschetz motive, and to the motive  $\mathbf{m}(X)$  being a mixed Tate motive over  $\mathbb{Z}$ . An important question in the ongoing investigations of the relations between quantum field theory and motives is understanding when (for which Feynman graphs) the varieties  $X_\Gamma$  (or equivalently  $Y_\Gamma$ ) are mixed Tate motives. This question has attracted a lot of attention in recent years.

We can define a max-plus character related to the behavior of the counting functions  $\#X_p(\mathbb{F}_q)$  for the graph hypersurface complement, that expresses the question of their polynomial countability.

**Lemma 8.6.** *Let  $\mathcal{H}$  be the Hopf algebra of Feynman graphs and let  $\psi : \mathcal{H} \rightarrow \mathbb{T}_{\max}$  be defined by  $N(Y_\Gamma, q) \sim q^{\psi(\Gamma)}$ , up to lower order terms in  $q$ , if  $Y_\Gamma$  is polynomially countable and  $\psi(\Gamma) = -\infty$  if it is not. Then  $\psi$  is a max-plus character, namely  $\psi(xy) = \psi(x) + \psi(y)$ .*

*Proof.* The counting function  $N(X, q) = N([X], q)$  factors through the Grothendieck ring, hence by (8.6) we have  $N(Y_\Gamma, q) = N(Y_{\Gamma_1}, q)N(Y_{\Gamma_2}, q)$  for a disjoint union  $\Gamma = \Gamma_1 \sqcup \Gamma_2$ . If both are polynomially countable, then the exponents of the leading terms satisfy  $\psi(\Gamma) = \psi(\Gamma_1) + \psi(\Gamma_2)$ . If at least one of them is not polynomially countable then  $\psi(\Gamma) = -\infty$ , which is also equal to  $\psi(\Gamma_1) + \psi(\Gamma_2)$ , since one of these terms is also  $-\infty$ . Thus, the result follows.  $\square$

The simplest possible Rota–Baxter operator of weight  $-1$  is the identity,  $T = id$ , which obviously satisfies the linearity hypothesis  $T(a + b) = T(a) + T(b)$  discussed in §7. Observe that, in the case where the operator  $T$  is linear (in the ordinary sense), the argument of Proposition 7.1 goes through unchanged, if we replace the min-plus tropical semiring with the analogous max-plus  $\mathbb{T}_{max}$  semiring, by simply replacing  $\oplus = \min$  with  $\oplus = \max$ , and  $\infty$  with  $-\infty$  as the additive unit.

In the case of the max-plus character of Lemma 8.6, the preparation  $\tilde{\psi}(x)$ , with respect to  $T = id$  then acquires a very simple geometric meaning. We have

$$\tilde{\psi}(\Gamma) = \max\{\psi(\Gamma), \tilde{\psi}(\gamma) + \psi(\Gamma/\gamma)\} = \max\{\psi(\Gamma), \sum_{j=1}^N \psi(\gamma_j) + \psi(\gamma_{j-1}/\gamma_j)\}$$

where the maximum is taken over all nested families of subgraphs  $\gamma_N \subset \gamma_{N-1} \subset \cdots \subset \gamma_0 = \Gamma$ . In the case of a graph  $\Gamma$  for which  $Y_\Gamma$  is not polynomially countable, the preparation  $\tilde{\psi}(\Gamma)$  extracts chains of subgraphs and quotient graphs that are polynomially countable.

A similar example is obtained by considering cases where the graph hypersurfaces  $X_\Gamma$  and  $Y_\Gamma$  depend on parameters (for example, if one works in the massive, instead of massless case, or if one considers the hypersurface defined by the second Symanzik polynomial, instead of the first, so that one has the dependence on the external momenta. In such cases the max-plus character  $\psi(\Gamma)$  defined above takes values in a semiring  $\mathbb{S}$  of functions on the set of parameters, with values in  $\mathbb{T}_{max}$ . One can then consider Rota–Baxter operators of weight  $-1$  given by multiplication by the characteristic function of certain subsets of parameters. The corresponding preparation, as in the simpler case above, would identify subgraphs and quotient graphs that are polynomially countable for specific choices of the parameters.

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